Hydraulic Servo Systems
Analysis and Design

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FOREWORD

In order to be useful, a book should be a good impedance match to its audience. Unfortunately audiences vary widely, and a presentation which attempts to satisfy too wide a range of readers often succeeds in being of little use to any of them. In the case of engineering books one excellent solution to the problem is to aim the book at the working engineer, and to add material of interest to the student or to the specialist only to the extent dictated by the circumstances of publication. Marcel Guillon is eminently qualified for this task.

Perhaps the strongest impression gained from reading both this book and its French predecessor is that they were written by a designer—by a man who has to use all of the tools at his disposal to create a product that can be made, that will perform and that can be depended upon. The structure of the book is the usual one. It starts by enunciating basic principles and proceeds to actual examples, going from the general to the specific and from the simple to the more complicated. This structure seems to be almost inevitable for this kind of book, perhaps because that is the way engineering is done. Unlike most books, however, this one is enlivened by a host of problems, comments, footnotes and qualifications derived from many year's experience as a designer and project manager, and supplemented by numerous appendices, graphs and other aids to both understanding and calculation. It is these supplements and side remarks that seem to me to be perhaps the most valuable feature of the work.

A book is unacceptable to both publishers and readers if it is too large, but the author always wants to include as much as possible. This typical engineering situation of conflicting requirements is usually solved by an engineering compromise: make the writing compact and concise. Unfortunately, this leads to difficulty of interpretation, and the optimum combination of size, content, ease of reading and freedom from ambiguity is a matter of personal taste.

Guillon's book suits my taste very well. Theorists may not like it because it minimizes the use of their specialized jargon and because it pays little attention to detailed proofs of the formulas given. Students (and their instructors) will find it heavy going for the same reason, although a competent instructor could use it as the basis for a marvellous course if he would supply the necessary explanations; many problems (without pat answers) are included and seem to me to be very well chosen. For the engineer in real life, who has a good general background, who wants to improve his competence in hydraulic servo design and who is willing to take a little trouble to do so, this book should be most valuable.

J. F. BLACKBURN

quickly attracted the attention of the aircraft industry and manufacturers of machine tools, testing machines, presses, rolling mills, machinery, etc.

Since I have been fortunate enough to take part in the first major French achievements in this field with the Air Equipement branch of the Ducellier–Bendix Air Equipement Co., for example, a photograph and description of the electrohydraulic control system for the second stage of the 'Diamant' rocket will be found in Chapter 9, I decided to write two new Chapters (8 and 9), dealing mainly with electrohydraulic servo systems and illustrated with examples taken from recent designs.

I would like to thank the directors of Sud Aviation and the Société pour l'Etude et la Réalisation d'Engins Balistiques who kindly gave me permission to include discussions of the Super Caravelle elevator control and the control system for the 'Diamant' second stage.

I also wish to thank Dr. J. F. Blackburn who has been a source of encouragement and who was kind enough to introduce my book; Mr. R. T. Griffiths who performed the lengthy task of translation; and Dr. O. N. Lawrence and Messrs. C. W. Sully, F. Brooks, P. Davies, W. N. Brainbridge, D. Parsons and J. M. Lawrence for their help with the final checking of the English text.

M. Guillon

PREFACE TO THE FIRST ORIGINAL EDITION

This book has been written for those who design and use hydraulic systems and, indeed, for any reader who is interested in this comparatively little known subject. It is based on reports and notes which I have compiled but not previously published in the course of many years' work on aircraft hydraulic systems. It is not a theoretical treatise which claims to clarify or re-examine any particular field of fluid mechanics, nor is it a technological catalogue with detailed descriptions and limited scope. It contains the general laws and methods of analysis which are already known but not as yet set down in any one textbook. These laws and methods are illustrated by numerous examples, taken from actual industrial problems, with numerical data corresponding to modern hydraulic equipment. The material is presented in such a way as to make it useful to the practical engineer.

The analysis of hydraulic circuits is supposedly difficult because of the non-linearity of the associated equations. It is rare, however, that these equations cannot be easily and validly linearized in the neighbourhood of an operating point. It is therefore natural to divide this book into two main parts, corresponding to the two successive stages in the analysis of a hydraulic system.

The first part, called 'Static Performance', deals with the determination of steady-state conditions.

The second part, called 'Dynamic Performance', deals with the transient behaviour of the system about the steady-state condition, i.e. essentially problems of stability, response speed and accuracy.

There is a third part which contains suitable examples illustrating the application of the methods of analysis derived in the first two parts.

The problems encountered in hydraulics can generally be divided into a limited number of groups, regardless of the type of equipment involved, and these groups are dealt with in separate Chapters. This method of presentation has its disadvantages. The more experienced reader may be annoyed to find certain apparently obvious points dealt with in too much detail. These seemingly superfluous Sections, however, have been included at the request of engineers who were kind enough to read the original manuscript and even to put it to the test as a handbook in everyday use. For this reason, a number of diagrams and equations have been repeated in the practical examples, in order to save time and to avoid possible mistakes. There is also a certain degree of elaboration on general points such as the concept of loss of head, a section on forces and stiffness, etc. which the reader may consider as out of place, but which provide an opportunity of revising the basic concepts involved in the later analyses.

The utilitarian aim of this book also results in a certain lack of order and harmony in its presentation. In any experimental work, an extremely broad knowledge is necessary to enable the engineer to tackle the different types of problems which were mentioned above. Research work has been carried out on these problems for some time with a certain amount of success. Thus an efficient hydraulic engineer has to apply both old empirical methods and new mathematical theories when faced with a design problem for a certain type of equipment. This broad knowledge is even more necessary in hydraulics than in other branches of engineering, and for this reason it is a difficult but interesting subject. This difficulty has not been suppressed or glossed over in the presentation of this book.

I hope the reader will not be deterred by these facts and will find the following pages interesting. I hope that the book will prove to be a useful tool for the engineer to help him to solve his everyday problems and that it will demonstrate the simplicity and effectiveness of the analysis of the dynamic performance of hydraulic systems. Finally, I hope that the non-specialist reader will learn something about hydraulics, its recent developments and its possibilities.

I would like to acknowledge and thank the following associates whose help has been invaluable in the preparation of this book: the very competent engineers under whose direction I have been privileged to work: Messrs. Szydlowski and René Leduc; the authors responsible for introducing modern methods of hydraulic analysis to France, whose books are quoted in the following pages, the engineers whose advice and assistance I found very helpful, particularly Messrs. Bernascon, Reynaud, Mathieu, Morisset, Pascal, Faillat, Blondel, Poncet and Mary, and the directors of the Ducellier–Bendix Air Equipement Co. for their permission to reproduce most of the examples presented.

Finally, I would be grateful to hear readers' comments on this book and will be glad to correspond on any subject.

CONTENTS

PART I. STATIC PERFORMANCE

As explained in the Preface, this book consists of two main parts which deal with the static and dynamic performances of hydraulic systems.

The term 'static', which is normally used, is not strictly accurate. It refers to the steady-state operating conditions, and the static analysis does not consider whether these conditions can exist physically, i.e. whether the system is stable, nor does it consider the way in which the steady-state conditions are established (transient regime).

1

ESTIMATION OF LOSS OF HEAD

1.1. THE CONCEPT OF LOSS OF HEAD

When a *perfect fluid* flows in a hydraulic circuit there is *no loss of energy*.

With the exception of *active components*, i.e. components where there is an exchange of energy between the system and its surroundings (e.g. pumps and turbines), the total energy in the fluid (cf. Chapter 2) under conditions of steady flow is constant. This total energy is normally quoted per *unit volume* of the fluid. It has the dimensions of pressure

$$\left(\frac{ML^2T^{-2}}{L^3} = ML^{-1}T^{-2} \qquad \text{or} \qquad \frac{FL}{L^3} = FL^{-2}\right)$$

and is called the *total pressure*, P_t.

Sometimes it is divided by the *weight* of the fluid. It has then the dimensions of length

$$\left(\frac{ML^2T^{-2}}{MLT^{-2}} = L \qquad \text{or} \qquad \frac{FL}{F} = L\right)$$

and is called the *total head*, H_t.

For incompressible fluids, the total head is related to the total pressure by the equation

$$H_t = P_t/\varrho g\ ,$$

where ρg is the specific weight.

In the flow of a *real fluid*, work has to be done against the viscous forces of friction, and there must be a corresponding *loss of energy*. (This lost energy is transformed to heat energy which is absorbed by the fluid and its surroundings.)

In a passive component, this loss of energy must result in a reduction of the total energy of the fluid.

Between two cross-sections, 1 and 2, in an elementary stream tube there will be a *reduction in the total energy, known as the loss of head*. According to whether we are considering total pressure or total head, the loss of head is given by

$$\Delta P = P_{t_1} - P_{t_2} \qquad \text{or} \qquad \Delta H = H_{t_1} - H_{t_2}$$

and will have the dimensions of pressure or length.

When considering low-pressure circuits, and especially flows which are open to the atmosphere and where the free surface level can be measured, we can use the total head system. In this book, however, we are mainly concerned with high-pressure circuits and will use the total pressure concept, since the height of the head at any point is not materialized by an actual level being obtained. *Hence the loss of head will always have the dimensions of pressure.*

Since we are dealing with liquids, which have *low coefficients of compressibility,* we will define the loss of head by the general form of Bernoulli's equation, as shown in Appendix 1.1:

$$\boxed{\Delta P_{1-2} = P_{t_1} - P_{t_2} = (p_1 - p_2) + \rho n g\,(h_1 - h_2) + \frac{\rho}{2}(V_1^2 - V_2^2)} \qquad (1)$$

where

ρ = density of the fluid

g = acceleration due to gravity

n = load factor

(ng = acceleration to which the circuit is submitted; e.g. in a manoeuvering aircraft, in a revolving engine or in a system which is subjected to centrifugal force)

p_1, p_2 = static pressures at sections 1 and 2

V_1, V_2 = velocities of flow at sections 1 and 2

h_1, h_2 = heights of sections 1 and 2 above a horizontal reference plane (or a reference plane perpendicular to the direction of acceleration if it is not due to gravity).

Note 1—For high-pressure circuits of limited size, the term $\rho ng(h_1 - h_2)$ is always small compared with the other two terms. It can therefore be neglected and the equation written

$$\Delta P_{1-2} = P_{t_1} - P_{t_2} = (p_1 - p_2) + \frac{\rho}{2}(V_1^2 - V_2^2) \qquad (2)$$

The term $\rho V^2/2$ is known as the '*dynamic pressure*'.

This equation states that the loss of head between two points in a stream tube is equal to the algebraic sum of the drop in static pressure and the drop in dynamic pressure between the points.

Note 2—The definition of loss of head given above is valid for a stream tube of infinitely small cross-sectional area. If the flow can be considered as being made up of a number of stream tubes, each of which has the same total pressure at sections 1 and 2, then the loss of head between these sections can be determined without difficulty. This is usually the case for flow in pipes.

The experimental determination of the loss of head depends on the measurement of *static pressures*, and in order to obtain accurate results it is necessary
(1) to make allowance for velocity ($\rho V^2/2$ terms);
(2) to measure the pressure either in a straight length of pipe at a point several diameters away from the nearest bend or obstruction (so that the velocities and pressures are constant) or at a position where the cross-sectional area is large enough for the speed of flow to be very small. Measurements should not be made at sections where the flow is disturbed.

1.2. THE DIFFERENT TYPES OF FLOW

1.2.1. THE EXISTENCE OF DIFFERENT TYPES OF FLOW

The main difficulty in determining the loss of head arises from the fact that different types of flow can exist within the circuit, each with different values of the loss of head, and it is not always easy to determine which of these will be present in practice.

If we neglect molecular flow, which is governed by complex laws and which is found only in passages of extremely small physical size such as that formed by placing two polished pieces of metal in contact with each other, or in the flow through porous media, we will be concerned with *laminar* and *turbulent* flow.

1.2.2. LAMINAR FLOW

In *laminar* flow (lamina = thin plate), neighbouring fluid particles move with parallel velocity vectors. The flow can be divided into a number of small individual stream tubes between which there is no exchange of fluid.

Owing to the viscosity (degree of stickiness) of the fluid each stream filament tends to slow down its faster moving neighbour and as a result of this the velocity will be zero at the walls and will increase with the distance from the wall.

It is shown in Appendix 1.2 that the frictional force is proportional to the coefficient of viscosity and to velocity gradient. Thus, *for laminar flow, the loss of head will be proportional to the coefficient of viscosity and the rate of flow of the fluid.*

1.2.3. TURBULENT FLOW

In turbulent flow, the fluid particles deviate from the mean path with a disorderly motion and eventually, due to viscosity, their kinetic energy (proportional to V^2) is dissipated as heat. These deviations are facilitated by local irregularities in the circuit (abrupt variations in the cross-section, bends, etc.) and by the state of the walls (roughness).

One can therefore expect that the loss *of head in turbulent flow will be relatively independent of the coefficient of viscosity but proportional to the square of the rate of flow and a function of the shape and roughness of the walls.*

1.2.4. CONDITIONS NECESSARY FOR LAMINAR AND TURBULENT FLOW

The theory of fluid flow shows, and practical experiments have confirmed, that laminar flow in a hydraulic circuit can be made turbulent by

(*a*) increasing the speed of the flow

(*b*) reducing the coefficient of viscosity of the fluid

(*c*) increasing the geometrical dimensions of the duct while keeping the *velocity of flow constant* (i.e. the volume flow increasing with the square of the linear dimensions*.

* This must not be confused with the increase of the geometrical dimensions of a component in a given circuit of constant volumetric flow, which gives the opposite result, since the velocity will decrease.

More precisely, with a length, L, defining the scale of the duct, the velocity, V, of the flow at a clearly defined point in the circuit, and the kinematic viscosity, ν, of the fluid (see Appendix 1.2 and Chapter 11, Graphs A and B), we can form the dimensionless quantity

$$\boxed{Re = \frac{VL}{\nu}}$$

called the Reynolds number*. There exists a value of this number called the critical Reynolds number, Re_c, such that if

$\mathrm{Re} < \mathrm{Re}_c$, *the flow will be laminar*
$\mathrm{Re} > \mathrm{Re}_c$, *the flow will be turbulent*

For flow in pipes, taking the internal diameter D as the length L in the definition of Reynolds number and the mean velocity (i.e. the volume flow divided by the

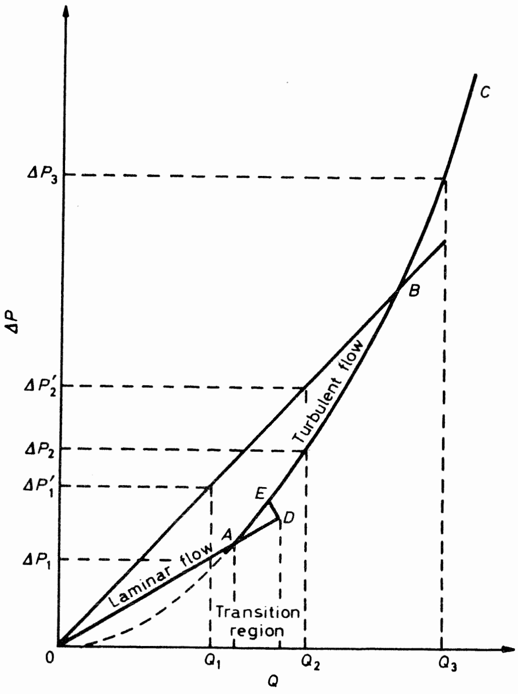


Figure 1.1. Variation of loss of head with rate of flow in a pipe.

If the transition region is ignored, the ΔP–Q curve for a fluid of kinematic viscosity ν_1 consists of a straight line, OA (laminar flow) together with a segment of a parabola, ABC (turbulent flow).

For a kinematic viscosity $\nu_2 > \nu_1$ (for example, the same fluid at a lower temperature), the curve consists of a straight line, OB (laminar flow) together with a segment of the parabola BC (turbulent flow).

When the viscosity changes from ν_1 to ν_2, the loss of head increases if the flow is initially laminar (flow Q_1) or if it becomes laminar (flow Q_2), but remains constant if initially turbulent flow remains turbulent.

cross-sectional area) as the velocity, V, *the value of the critical Reynolds number is about* 2,500.

In more complex components, there is no well defined transition between laminar and turbulent flow. Both types of flow can coexist in different parts of the component.

* See note, p. 27.

The definition of Reynolds number introduces the problem of the choice of L and V. This will be considered later when dealing with individual components.

Note 1—The value of 2,500 given for the critical Reynolds number is not fixed and definite like those given later for various components. For example, in the laboratory, by increasing the speed of flow very carefully, laminar flow can be retained for Reynolds numbers far greater than 2,500, reaching 5,000 or even 6,000.

Although transition from laminar to turbulent flow may be delayed until the Reynolds number is relatively high, if laminar flow is to be obtained again by decreasing the flow velocity, the Reynolds number must be lowered to below 2,500 before this occurs.

The laminar flow retained beyond the critical Reynolds number is therefore an unstable flow (cf. the phenomenon of superfusion) (see *Figure 1.1*).

Note 2—The two types of flow are easily distinguished at the outlet of a water tap. The flow is laminar when the tap is slightly open (transparent flow) and turbulent when the tap is fully open (more opaque flow).

1.3. A PRACTICAL METHOD OF ESTIMATING LOSS OF HEAD

There are three types of problems:
(1) finding the loss of head for a given volume flow through a given component;
(2) finding the volume flow through a given component with a given loss of head;
(3) finding the dimensions of a component for a given volume flow and given loss of head.

The loss of head can be divided into two types: (*a*) *localized losses of head*, which develop in irregularities in the circuit (junctions, pipe unions, elbows, changes in section, nozzles, etc.), and (*b*) *distributed loss of head* which is developed along the pipes.

In addition, we have the two different types of flow, laminar and turbulent.

In the circuits which we shall be considering, turbulent flow is by far the most common. That is why, with the exception of a few cases shown later in Sections 1.5.3 and 1.6, we will relate loss of head to dynamic pressure and will write it in the form

$$\boxed{\Delta P = \xi \frac{\varrho}{2} \frac{Q^2}{S^2}} \tag{3}$$

where

ρ = density of the fluid
Q = volume flow
S = reference cross-sectional area
ξ = coefficient of loss of head of the component considered
($Q/S = V$ = mean velocity in the reference section).

In these conditions it is only necessary to know ξ to calculate the loss of head. We shall show that the estimation of the coefficient ξ of a component is possible knowing the coefficient of a similar component.

This method is simple. The accuracy may not be high (about 25 or 50 per cent) in complex components, but these are usually connecting components with relatively small losses of head compared with the overall circuit pressure, and usually no great accuracy is required.

On the other hand, the method is quite accurate (about 5 per cent) for restricting components (orifices, nozzles, valves) in which the loss of head is most important.

Note—In practice, it is usual to use specific weight, w, rather than density, ρ $w = \rho g$ (see Unit Conversion Tables, Chapter 11).
Eqn. (3) becomes

$$\Delta P = \xi \frac{w}{2g} \frac{Q^2}{S^2} \tag{3'}$$

With the industrial units
w, in g/cm^3
ΔP, in kg/cm^2
$Q/S = V$ in m/sec
eqn. (3) becomes:

$$\Delta P = 1{\cdot}02 \cdot 10^{-2} \xi \frac{w}{2} V^2 \tag{3''}$$

This equation is shown in Graph C which gives $\Delta P/\xi = f(V)$ for different values of w.

1.4. LOCALIZED LOSS OF HEAD

There are two types of components, *restricting* (orifices, nozzles, valves, etc.) and *connecting components* (junctions, elbows, etc.).

1.4.1. RESTRICTING COMPONENTS

Components whose function is to control the pressure of the flow are called restricting components. They have a high loss of head and the flow in them is usually turbulent.

If the loss of head is the unknown, turbulent flow is easily verifiable (see below). In other cases, the flow is assumed turbulent, and when the calculation is complete the validity of the assumption can be checked, even though in a few cases some modification of the calculation may be necessary.

1.4.1.1. *Turbulent flow through an orifice plate*

Let us consider the most simple restricting component: an orifice plate (see *Figure 1.2*). As the fluid passes through the orifice, it accelerates to maximum velocity, V'_R, and its static pressure decreases accordingly. This operation takes place with practically no loss of energy. On the other hand, the reconversion of the extra kinetic energy back into pressure energy depends largely on the downstream shape of the orifice. If there is no fairing downstream of the orifice, almost all the extra kinetic energy is lost and the total loss of head is therefore given by

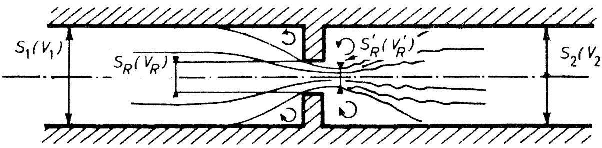
$$\Delta P = \frac{\varrho}{2}\left(V_R'^2 - V_2^2\right)$$

Since V_R' is much greater than V_2, we can neglect V_2^2 with respect to $V_R'^2$, giving

$$\Delta P = \frac{\varrho}{2} \cdot V_R'^2 \qquad (4)$$

If the upstream edge of the orifice is square, i.e. if there is no bevel, then the jet of fluid *contracts* as it passes through the orifice to give a jet of fluid smaller in cross-sectional area, S_R', than the cross-sectional area of the orifice itself, S_R.

Figure 1.2



We define a coefficient of contraction, $c = S_R'/S_R$. It is convenient to relate the loss of head in an orifice to the *actual geometrical area*, S_R. Thus, since by continuity, $V_R S_R = V_R' S_R' = Q$

$$\Delta P = \frac{1}{c^2}\,\frac{\varrho}{2}\,V_R^2 = \frac{1}{c^2}\,\frac{\varrho}{2}\,\frac{Q^2}{S_R^2}$$

or, if we put $1/c^2 = \xi$ (cf. eqn. 3)

$$\boxed{\Delta P = \xi\,\frac{\varrho}{2}\,\frac{Q^2}{S_R^2}} \qquad (4')$$

This result shows that, when the coefficient ξ is related to the area of the orifice, it can often be greater than unity.

For an orifice whose diameter is of the same order of magnitude as the thickness of the plate, if the edges are square, i.e. no bevel or chamfer (*Figure 1.3a*):

$$\xi = 1{\cdot}7 \text{ to } 1{\cdot}9^*$$

If the upstream edge of the orifice is bevelled (*Figure 1.3b*), the value of ξ is lowered to about 1 so that $S_R' \to S_R$.

If, in addition, the downstream edge is bevelled (*Figure 1.3c*), provided that the equivalent conical angle is less than about 10°, this lowers ξ still further,

* Some books give much lower values, $\sim 1{\cdot}5$, but these do not correspond to the author's measurements.

since it effectively constitutes a diffuser which allows efficient recovery of the kinetic energy.

But the efficiency of this energy recovery depends on so many parameters that the best way to make a restricting orifice is to drill a small hole whose diameter

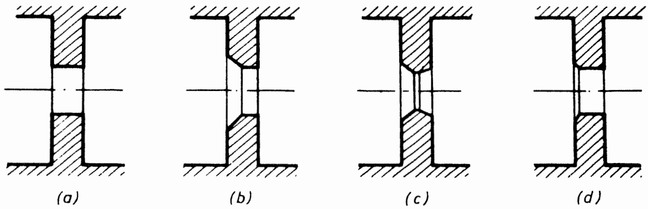


Figure 1.3

is about the same as the thickness of the plate, smoothing off any burrs at the edges, and to allow for losses by putting

$$\xi = 1{\cdot}8$$

Graph D (Chapter 11) gives

$$\Delta P = \xi \frac{w}{2g} \frac{Q^2}{S^2}$$

for a circular orifice of diameter ϕ (mm) with $\xi = 1{\cdot}8$ and $w = 0{\cdot}8$ g/cm^3 (kerosene).

To obtain good accuracy, one can, by starting with an orifice slightly undersized (a few per cent on diameter), adjust the loss of head to the required value by progressively *chamfering the upstream edge* (*Figure 1.3d*).

1.4.1.2. *Other restricting components*

For other restricting components (valves, taps, etc.) the coefficient ξ can be estimated by taking the *minimum cross-sectional area* and considering the equivalent squared-edged circular orifice having the same area.

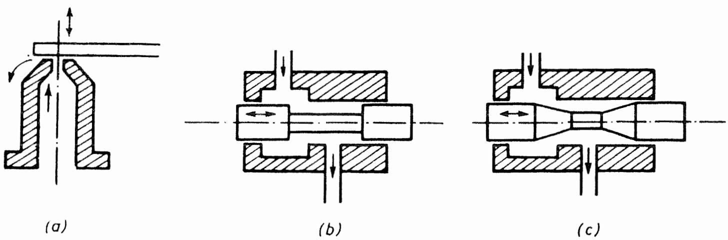


Figure 1.4

Note that values of ξ less than 1 are very rare. For the components shown in *Figure 1.4*, the values of ξ are

Flapper-and-nozzle valve (*Figure 1.4a*)	1·6–1·9
Control valve with square-edged spool (*Figure 1.4b*)	$\simeq$1·6
Control valve with conical spool surfaces (*Figure 1.4c*)	1–1·6

1.4.1.3. *Laminar flow*

When the flow is very slow, the loss of head due to the eddy currents and turbulence downstream of the orifice is very small compared with that arising from friction at the walls (even for very sharp-edged orifices) since the former decreases with V^2 and the latter with V.

If we continue to use $\Delta P = \xi \,.\, \rho/2 \,.\, Q^2/S^2$ (eqn. 3), the coefficient ξ should therefore increase. This effect takes place when the Reynolds number, $\mathrm{Re} = VD/\nu$, is decreased (where V is the mean velocity in the narrowest section, D the diameter or the equivalent diameter* if the orifice is not circular, and ν the kinematic viscosity).

This phenomenon (of ξ increasing) is progressive and it is difficult to define a critical Reynolds number. Graph E shows the experimental results for the variation of ξ with Re for different orifices with diameters about 1 mm, using a standard hydraulic fluid.

From these graphs and from other less accurate measurements the following result has been obtained.

The coefficient of loss of head for an orifice of the type shown in *Figure 1.3a* with laminar flow is given by the equation

$$\xi = \frac{64}{Re}\left(\alpha + \frac{L}{D}\right)$$

where L and D are the length and diameter of the hole, respectively, and α is a constant whose value† is about $\frac{2}{3}$.

If the transition from laminar to turbulent flow is well defined, knowing that the value of ξ is about 1·8 for turbulent flow, we have

$$Re_t = \frac{64}{1{\cdot}8}\left(\alpha + \frac{L}{D}\right) = 35\left(\alpha + \frac{L}{D}\right)$$

where Re_t is the Reynolds number at transition (for example, $\mathrm{Re}_t = 58$ when $L = D$).

In practice, the *transition region* is not well defined but extends between about $\mathrm{Re}_t/4$ and $4\mathrm{Re}_t$, i.e. from $\mathrm{Re} = 9(\alpha + L/D)$ to $140(\alpha + L/D)$; if $L = D$, this is from $\mathrm{Re} = 15$ to $\mathrm{Re} = 233$.

In this region, ξ decreases from about 7·1 to about 1·8. The following list gives approximate values of ξ for various Reynolds numbers (see Graph E).

$\dfrac{Re}{\alpha + L/D}$	9	12	15	20	30	50	70	100	140
ξ	7·1	6	5·5	4·6	3·8	3	2·5	2·1	1·8

* The 'equivalent diameter' is that of a circular passage for which the cross-sectional area of flow divided by the wetted perimeter would be the same as the duct under consideration, i.e. $D = 4S/P$, where S is the cross-sectional area flow and P the wetted perimeter.

This operation should not be extended to very irregular sections, in particular to those where one dimension is very much smaller than the other (see Section 1.6.1).

† More accurate measurements are needed to determine α more precisely. It will probably not remain absolutely constant, but the value of $\frac{2}{3}$ has given good results in practice.

It must be remembered that

(1) laminar flows are very rare in restricting components. To obtain laminar flow, the Reynolds number must be extremely low, and in any case the Reynolds number in an orifice is greater than in the pipe itself, since the product $V \,.\, D$ increases when D decreases with constant volume flow Q.

(2) when a restricting component with a linear loss of head is required, it is usual to use either a capillary tube (Section 1.5.3)—but then the coefficient of loss of head varies with the viscosity—or a special component (see Section 3.2.2.2).

1.4.2. CONNECTING COMPONENTS

Included in this category are bends, albows, junctions, non-return and flow control valves in a fully open position, etc. For these items the only useful reference area is the inside area of the inlet tube, and this will be called the *nominal area.*

1.4.2.1. *Turbulent flow*

The value of the coefficient ξ for these components varies to a surprisingly small extent considering the differences in type and construction.

The following list gives typical values:

Component	ξ
Straight joints (or straight across the top of a T junction)	0·5
Elbows	1·0
Normal banjo unions (cf. Graph F)	2–3
Special banjo unions (Quinson type)	1·5
Valves, etc.	3–6

For accurate values of ξ it is obviously necessary to know the actual size and shape of the component in more detail. This is discussed at length in more specialized books[1].

It is a pity that these connecting components have such high values of the coefficient ξ. The reason for this is that a joining component is efficient only if it has no cross-sectional area which is smaller than its nominal area. It can be seen, therefore, that valves of the 'integral passage' type whose coefficients ξ are about 5, could have their loss of head halved by making a few minor alterations to the design.

1.4.2.2. *Laminar flow* (cf. Chapter 10, Example 1)

Connecting components can be considered as being intermediate between pipes and restricting components.

Defining Reynolds number by

$$Re = \frac{VD}{\nu}$$

where V is the velocity in the *nominal* area and D the nominal diameter, it has been found that the critical value is generally between 1,000 and 3,000, as shown in Graph F.

As Reynolds number is reduced, the onset of laminar flow is more clearly defined than for restricting components but less clear than for pipes (cf. Graphs E and G).

Note—The choice of the nominal area as the reference area seems to complicate the definition of Reynolds number. Consider two components which are identical except that they are made to fit different sizes of pipe and will therefore have different nominal areas. Critical flow will occur in both components at the same volume flow. Now, if the ratio of the diameters of the two pipes, P and P', is λ, their Reynolds numbers will be in the ratio

$$\frac{Re'}{Re} = \frac{V'D'/\nu}{VD/\nu} = \frac{(V/\lambda^2)\cdot\lambda D/\nu}{V\cdot D/\nu} = \frac{1}{\lambda}$$

The critical Reynolds number decreases as the size of the pipe is increased.

But increasing the size of the pipe for a fixed component means that the component effectively becomes under-dimensioned.

It must be remembered, however, that the critical Reynolds number, as defined above, is lowered as the component becomes under-dimensioned, and in practice it remains within the limits quoted.

Note 2—Laminar flow is much more common in connecting than in restricting components. Consider a restricting component of diameter D_R in series with a connecting component of nominal diameter D_C. Obviously $D_C > D_R$, and if we put $D_R/D_C = \lambda$, we have, as in Note 1, $\mathrm{Re}_C/\mathrm{Re}_R = 1/\lambda$, i.e. the Reynolds number for the connecting component is less than that of the restricting component.

In view of this it is advisable to check that the flow in the connecting components is in fact turbulent. If it is not, then the flow will not be turbulent in the adjoining pipes either, and for the purposes of an approximate laminar flow calculation of the losses, the component can be replaced by an equivalent length of piping (see Section 1.5).

1.5. DISTRIBUTED LOSS OF HEAD

1.5.1. General Equation

In pipes where the length is greater than the diameter, the loss in head is proportional to the length; this type of loss is called *distributed loss of head.*

In order to keep a dimensionless coefficient of loss of head, ξ, while taking into account the fact that the loss is proportional to length, it is usual to write

$$\boxed{\xi = \lambda \frac{L}{D}} \tag{5}$$

where L is the length of the pipe, D its diameter and λ the *coefficient of distributed loss of head* or *friction factor.*

Substituting this into the general equation $\Delta P = \xi \cdot \rho/2 \cdot Q^2/S^2$ gives

$$\boxed{\Delta P = \lambda \frac{L}{D} \frac{\varrho}{2} \frac{Q^2}{S^2}} \qquad \textbf{(6)}$$

and the calculation of loss of head is reduced to the estimation of λ.

The reference area is obviously the *cross-sectional area of the pipe.*

Note—For short pipes, the loss of head at the inlet becomes of appreciable magnitude when compared with the distributed friction losses, so that we have

$$\Delta P = \left(\lambda \frac{L}{D} + \xi_e\right) \frac{\varrho}{2} \frac{Q^2}{S^2} \qquad (7)$$

where ξ_e is the coefficient of localized loss of head at the entry.

The value of ξ_e is about 0·5–1 for straight pipes (for further details, see Graph F).

The coefficient λ depends on a number of parameters, in particular on the Reynolds number, $\mathrm{Re} = VD/\nu$, and the relative roughness of the tube, $R_u = K/D$, where K is the mean height of the surface asperities*.

This subject is covered by most textbooks dealing with mechanics of fluids, and the principal results are shown in Graph F which gives values of λ as a function of Reynolds number for different values of the surface roughness. For the calculations given here, however, the use of Graph F is not necessary, since a high degree of accuracy is not required. In any case, it is difficult to assess the roughness of commercial piping. The approximation given below is sufficiently accurate.

1.5.2. TURBULENT FLOW

In the hydraulic systems used in aircraft, cars or machine tools the internal surfaces of the pipes are usually in good condition.

Without much error, we can use

$$\lambda = 0{\cdot}025$$

for all turbulent flows, i.e. *flows where the Reynolds number exceeds* 2,500. This approximation, together with the method for estimating localized losses of head (Sections 1.4.1.2 and 1.4.2.1), makes the evaluation of the loss of head in a circuit a quick and simple process. We have

$$\Delta P = \left[\Sigma\, \xi_c + 0{\cdot}025\ \Sigma \left(\frac{L}{D}\right)\right] \frac{\varrho}{2} \frac{Q^2}{S^2}$$

where ξ_c is the coefficient of loss of head of connecting components and L is the length of piping.

---

* Consider a surface made up of particles stuck to a smooth pipe with the spacing between the particles of the same order of magnitude as their mean diameter. If this pipe is hydraulically similar to the pipe under consideration, then K is the mean diameter of the particles. In practice, K is equal to mean height of the surface irregularities, provided they are sufficiently close and rough.

Graph G shows how the loss of head, per metre of piping, varies with the volume of flow, for different values of the pipe diameter, using $w = 0{\cdot}8$ g/cm^3.

Note 1—An alternative way of stating the approximate equation for the loss of head in pipes, which is easier to remember, is:

The loss of head for turbulent flow in pipes is equal to the dynamic pressure for each length of pipe $L = 40D$.

For this length,

$$\frac{\lambda L}{D} = \frac{0{\cdot}025 \times 40D}{D} = 1$$

Note 2—If there is a length, L', of pipe, diameter D', in series with a pipeline of diameter D, it can be considered as equivalent to a pipe of the same diameter but of length $L^* = L'(D/D')^5$. To check this, we can calculate the loss of head in each case.

For the length, L', of diameter D'

$$\Delta P' = \lambda \frac{L'}{D'} \frac{P}{2} \frac{Q^2}{S'^2} = \frac{1}{(\pi/4)^2} \lambda \frac{P}{2} \frac{L'}{D'^5} Q^2$$

For the length L^* of diameter D

$$\Delta P = \lambda \frac{L^*}{D} \frac{P}{2} \frac{Q^2}{S^2} = \frac{1}{(\pi/4)^2} \lambda \frac{P}{2} \frac{L^*}{D^5} Q^2$$

and since $\Delta P' = \Delta P$, $L^* = L'(D/D')^5$.

This is only true, however, provided that the flow is turbulent in both pipes. This is obviously the case if $D' < D$ since, as we have seen, the Reynolds number increases if the pipe size is decreased for a given volume flow.

Note 3—For pipes of small diameter, it is necessary to increase the value of λ slightly. Typical values would be:

$$\lambda = 0{\cdot}030 \quad \text{for } D = 4 \text{ mm}$$
$$\lambda = 0{\cdot}040 \quad \text{for } D = 2 \text{ mm}$$

1.5.3. LAMINAR FLOW

The laws governing laminar flow in pipes were discovered by Poiseuille while studying the circulation of the blood. These laws are established in Appendix 1.3. The theory is confirmed by experiment, so in calculations we can use the value of λ given by Poiseuille

$$\boxed{\lambda = \frac{64}{Re}} \tag{8}$$

We can see, therefore, that the loss of head in laminar flow does not depend on the roughness of the pipe. It varies linearly with the viscosity of the fluid ($\mathrm{Re} = VD/\nu$). Exact evaluation of the loss of head is therefore very easy.

In practice, however, it is not quite so straightforward since the coefficient of viscosity is not constant. It varies considerably with temperature (see Graph A), and allowance must be made for this in even an approximate estimate.

For systems designed to operate in cold surroundings, the loss of head should be calculated for the minimum temperature.

In practice, the dimensions of pipelines in hydraulic circuits are such that laminar flows are only present occasionally and at temperatures near the minimum.

Practical calculations (cf. Chapter 10, Example 1)—Eqn. 8, $\lambda = 64/\mathrm{Re}$, is convenient to use only when the unknown factor is the loss of head, i.e. volume flow and circuit dimensions are given. When we need to find either the flow or the dimensions for a given loss of head, it is preferable not to use the general expression given in Section 1.3 but to write directly (cf. Appendix 1.3):

$$Q = \frac{\pi}{128} \frac{1}{\nu\rho} \frac{D^4}{L} \Delta P \tag{9}$$

where

ν = kinematic viscosity
ρ = density
L = length of the pipe
D = diameter of the pipe
Q = volume flow

In practical units, eqn. (9) becomes

$$Q = 2{\cdot}41 \times 10^6 \frac{1}{\nu w} \frac{D^4}{L} \Delta P \tag{10}$$

or, rearranging terms

$$\Delta P = 0{\cdot}415 \times 10^{-6}\, \nu w \frac{L}{D^4} Q \tag{10'}$$

in which L and D are in cm, Q in cm^3/sec, ν in centistokes (mm^2/sec), ΔP in kg/cm^2; w, the specific weight, is in g/cm^2 (the same numerical value as the specific gravity).

1.6. PARTICULAR PROBLEMS

For a certain number of particular problems, the methods defined above are too clumsy or even, in some cases, impossible to apply. We shall consider some of these.

1.6.1. DETERMINATION OF THE LEAKAGE FLOW THROUGH THE CLEARANCE BETWEEN PARALLEL PLATES

The most usual problem is the calculation of the rate of leakage under a known pressure difference. The theory for *laminar* flow between two parallel flat plates and for the flow in the space between two cylinders, concentric or not, is given in Appendices 1.4 and 1.5.

There is obviously no point in reducing the result to the general equation (3). We have, for laminar flow between

(*A*) *two parallel plates*:
$$Q = \frac{1}{12}\,\frac{1}{\nu\varrho}\,\frac{le^3}{L}\,\Delta P \tag{11}$$

(*B*) *two cylinders*:
$$Q = \frac{\pi}{96}\,\frac{1}{\nu\varrho}\,\frac{DJ^3}{L}\,\alpha\,\Delta P \tag{12}$$

where

J = diametral clearance
ϵ = eccentricity (distance between the two axes)
α = coefficient of eccentricity

$$\left[\alpha = 1 + \frac{3}{2}\,\frac{\epsilon^2}{(J/2)^2}\right]$$

(Note that α varies from 1 to 2·5 when ϵ varies from 0 to $J/2$).
With the units

$$\left.\begin{array}{l} L = \text{length in the direction of flow}\\ \left.\begin{array}{l} l = \text{width of flow}\\ e = \text{thickness of flow}\end{array}\right\}\text{case } A \\ \left.\begin{array}{l} D = \text{mean diameter}\\ J = \text{diametral clearance}\\ \epsilon = \text{eccentricity}\end{array}\right\}\text{case } B\end{array}\right\}\text{in cm}$$

Q = volume flow, in cm^3/sec
ΔP = pressure difference, in kg/cm^2
w = specific weight, in g/cm^3 (same numerical value as density)
ν = kinematic viscosity, in centistokes (mm^2/sec)

these equations become

(*A*)
$$Q = 8{\cdot}18 \times 10^6\,\frac{1}{\nu w}\,\frac{le^3}{L}\,\Delta P$$

(*B*)
$$Q = 3{\cdot}21 \times 10^6\,\frac{1}{\nu w}\,\frac{DJ^3}{L}\,\alpha\,\Delta P$$

Note that
(1) these equations are valid only for laminar flow. If the clearance is very small or very large, the flow will be either molecular or turbulent, respectively, and thus the above equations will not hold. But molecular flow is to be found only between two mating polished surfaces, such as those in a tight metallic joint. Turbulent flows are rare and their presence is of little importance since, for a given pressure difference, there will be a lower rate of flow than for laminar flow (see *Figure 1.1* and Section 1.2.4). The leakage will then have been overestimated.
(2) The approximate method of calculation where the annular space is considered as equivalent to a circular tube of equal cross-sectional area to that of the annulus is completely erroneous, as shown at the end of Appendix 1·5.

(3) Experimental results do not always exactly confirm the theory for laminar flow. The leakage flow may be effectively constant while the parts are in relative motion, but diminish or sometimes even cease when there is no movement.

Clogging is often not sufficient to explain this phenomenon, and several authors have suggested that the cause may be electrostatic.

Sometimes, too, for a very narrow space, the flow across a given clearance can be different for two liquids having the same viscosity, which suggests that new parameters should be introduced into the theory.

The equations given above are valid for all cases, since they give the upper limit of the flow.

1.6.2. LOSS OF HEAD IN FILTERS

Filters consist of a large number of holes or channels of a given size placed in parallel and in series along the path of the liquid. They will therefore obey the laws defined above.

In practice, except for certain metallic filters with large meshes used with high loss of head, the flow is laminar.

A filter made up of N elementary circular channels of diameter d and length l, placed in parallel, will have a loss of head given by [cf. eqn. (9)]

$$\Delta P = \mu \frac{1}{N} \frac{128}{\pi} \frac{l}{d^4} Q \tag{13}$$

where μ is the coefficient of absolute viscosity: $\mu = \nu\rho$ (see Appendix 1.2).

Very often the elementary channels are neither regular nor exactly identical. It is only known that the number of channels is proportional to the area of the filtering surface, and so it is preferable to relate the loss of head to this area S, while including the parameters l and d in a constant relating to the type of filter under consideration.

Thus the general expression for the loss of head in filters is

$$\Delta P = K \frac{\mu Q}{S} \tag{14}$$

where K is a coefficient given by the manufacturer (its dimension is $1/L$).

For a particular type of filter, the manufacturers often use a coefficient $K' = K/S$, i.e. they include the surface area in the constant. In this case, eqn. (14) becomes

$$\Delta P = K' \mu Q \tag{14'}$$

where K' has the dimensions of $1/L^3$.

Sometimes the viscosity, μ, in the above equations is replaced by the kinematic viscosity, ν. This is fundamentally incorrect but, since the variation in density of the filtered liquid is very small, it involves no great error. Thus we have

$$\Delta P = K_1 \frac{\nu Q}{S} \tag{15}$$

or

$$\Delta P = K_1' \nu Q \tag{15'}$$

where K_1 and K_1' have the dimensions M/L^4 and M/L^6, respectively.

The numerical values of the coefficient K (or K', K_1, K_1') are extremely variable, not only with the degree of filtration obtained, as was shown in eqn. (13), but also with the physical nature of the filter element itself.

The following Table give values of K_1 in kg/(cm^3/sec . cS) for different materials, as quoted by the manufacturers.

Material	*Degree of filtration,* microns	K_1
Cardboard	5	0·12
Felt	80–100	0·025
Wire gauze	50	0·0025
	500	0·0012

The choice of filter is a compromise between the degree of filtration required, the space available and the length of time allowable between cleaning or changing the element.

In practice, when the degree of filtration is increased,

(1) the coefficient of loss of head increases;

(2) the pressure difference that the element is mechanically able to withstand is decreased;

(3) the tendency to clog up increases. For fine filters, it is advisable to design the element so that the loss of head through the filter is less than $\frac{1}{3}$ the pressure difference which would cause the rupture of the element.

In the present state of filter technology, the size of filter necessary is often prohibitive. This accounts for the efforts made by the users to limit the need for filtration by ensuring that the hydraulic fluid is clean before introducing it into the circuit; by differential filtering, the fine filtration being reserved for those small portions of the flow used in components where there is small clearance, such as regulation or control valves; by increasing the clearance for the flow wherever possible, and by rejecting otherwise attractive solutions which require the fluid to be too clean. The manufacturers, on the other hand, endeavour to produce very fine filter elements with small coefficients of loss of head and high mechanical strength.

Research is being made on:

(*a*) the use of gauzes made of synthetic fabrics; unfortunately, as mentioned above, the coefficient of loss of head tends to increase with time;

(*b*) the use of sintered metals; unfortunately they are very liable to clogging and even to blockage by superficial depression and compression of the particles;

(*c*) the use of ultra-thin metal sheets perforated by electrolytic or chemical means or, otherwise, very fine meshes produced by electrolytic deposition; the cost of manufacturing these elements is still very high.

Numerical example—Suppose a flow of 1,000 cm^3/sec (61 in.^{3}/sec) of standard hydraulic fluid at ambient temperature (ν = 25 cS) has to be filtered (*a*) at 5 microns and (*b*) at 50 microns.

(*a*) Using a cardboard filter for which $K_1 = 0{\cdot}12$ and $\Delta P_{\max} = 0{\cdot}2$ kg/cm^2 (2·85 lb./in.^{2}), and taking a clogging coefficient, $C = 5$

$$S = C\,\frac{K_1 \nu Q}{\Delta P_{\max}} = 5\,\frac{0{\cdot}12 \times 25 \times 1{,}000}{0{\cdot}2} = 75{,}000 \text{ cm}^2\ (11{,}700 \text{ in.}^2)$$

which is equivalent to assembling 750 standard elements of 100 cm^2 (15·6 in.^{2}) in parallel!

(*b*) Using a metallic filter for which $K_1 = 0{\cdot}0025$ and $\Delta P_{\max} = 1$ kg/cm^2 (14 lb./in.^{2}), and taking a clogging coefficient, $C = 3$

$$S = C\,\frac{K_1 \nu Q}{\Delta P_{\max}} = 3\,\frac{0{\cdot}0025 \times 25 \times 1{,}000}{1} = 188 \text{ cm}^2\ (29{\cdot}3 \text{ in.}^2)$$

i.e. two standard elements of 100 cm^2 in parallel are sufficient.

1.6.3. LOW-PRESSURE FLOW

If, in a flow where the pressure is low, there is an increase in flow velocity (due to a restriction or an orifice, for example), the corresponding reduction in static pressure may reduce the pressure level to a value equal to the vapour pressure of the liquid.

The pressure cannot be made lower than the vapour pressure. In fact, any attempt to reduce the pressure further succeeds only in vaporizing the liquid.

Up to now we have maintained that the flow in a hydraulic component depends only on the *pressure difference* across it. It must now be understood that, if the pressure upstream of a component is kept constant and the downstream pressure decreased, the flow will reach a maximum value when the pressure in some part of the component reaches the vapour pressure of the fluid. This phenomenon, called cavitation, is very important. Not only does it cause erosion and damage to the components themselves, but it is often responsible for anomalies in the operation of hydraulic circuits.

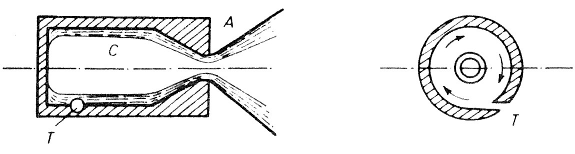


Figure 1.5

Cavitation must be avoided. It is likely to occur in aircraft hydraulic systems at altitude when the reservoirs for the fluid are not pressurized, and particularly *at the inlets of hydraulic pumps*.

This phenomenon is sometimes met with in high-pressure systems, for example in injectors. A vortex injector, made to atomize or spray liquids (injection of fuel in a jet engine), consists basically of a chamber, C, into which the liquid enters by one or more tangential orifices, T, and leaves through the axial nozzle, A (*Figure 1.5*). The liquid in chamber C has a high angular velocity which produces a radial pressure gradient. It can be shown that there is no liquid on the axis and that the nozzle does not run 'full'.

In certain cases, the operation of the injector depends not only on the pressure difference across it, but on the absolute value of the downstream pressure.

Note—When, as a first approximation, this effect is disregarded, the coefficient of loss of head of an injector can be calculated using the area of the hole (or holes) T as the reference area. Although this area is clearly less than that of the nozzle A, coefficients ξ in the order of 7 or 8 are obtained, which shows that it would be absolutely wrong to apply the methods given above to the holes and nozzle orifice in series.

1.7. CIRCUITS

The analysis of circuits in which the flow is completely laminar presents no difficulty, since it is analogous to the flow of current in an electrical circuit. But completely laminar flow is seldom encountered in practice.

In circuits where both laminar and turbulent flow exist, it is possible to make simple general rules. Mention will also be made of circuits composed of components where the flow is turbulent.

1.7.1. COMPONENTS IN SERIES

$$\Delta P_{tot} = \Sigma\, \Delta P = \Sigma \left(\xi \frac{\varrho}{2} \frac{Q^2}{S^2} \right) = \frac{\varrho}{2} Q^2 \Sigma \left(\frac{\xi}{S^2} \right)$$

$$(\xi/S^2)_{tot} = \Sigma\, (\xi/S^2)$$

If the reference areas of all the components are the same (connecting components)

$$\xi_{tot} = \Sigma\, \xi$$

If the components considered are restricting components of the same coefficient ξ, the equivalent area, S_e,is given by

$$1/S_e^2 = \Sigma\, (1/S^2)$$

This equation expresses a most important result: if several orifices (or their equivalents) are connected in series in a hydraulic circuit, and one of them is much smaller than the others, then the presence of the others can be neglected.

This fact is very useful when adjusting hydraulic resistance and is one of the reasons for the accuracy of hydraulic servo or regulating devices.

The hydraulic potentiometer—The hydraulic potentiometer consists of two orifices with variable cross-sectional area placed in series, so that the pressure between them can be varied. This device will be considered in Section 3.4.2.

1.7.2. COMPONENTS IN PARALLEL

The fundamental eqn. (3) can be written

$$Q = \sqrt{2/\varrho} \cdot S/\sqrt{\xi} \cdot \sqrt{\Delta P}$$

thus

$$Q_{tot} = \sqrt{2\, \Delta P/\varrho} \cdot \Sigma\, (S/\sqrt{\xi})$$

$$(S/\sqrt{\xi})_{tot} = \Sigma\, (S/\sqrt{\xi})$$

SUMMARY OF RESULTS

		Localized loss of head (Section 1.4)		*Distributed loss of head* (Section 1.5)	*Particular problems*	
Type of component		*Restricting components*	*Joining components*	*Pipelines*	*Clearance space between two cylinders*	*Filters*
Basic equation	fundamental units	$\Delta P = \xi \frac{\rho}{2} \frac{Q^2}{S^2}$		$\Delta P = \lambda \frac{L}{D} \frac{\rho}{2} \frac{Q^2}{S^2}$†	$Q = \frac{\pi}{96} \frac{1}{\nu\rho} \frac{DJ^3}{L} \alpha \Delta P$	$\Delta P = K \frac{\mu Q}{S}$
	industrial units*	$\Delta P = 10^{-2} \xi \frac{w}{2} V^2$		$\Delta P = 10^{-2} \lambda \frac{L}{D} \frac{w}{2} V^2$‡	$Q = 3 \cdot 2 \times 10^6 \frac{1}{\nu w} \frac{DJ^3}{L} \alpha \Delta P$	$\Delta P = K \frac{\nu Q}{S}$
Reference area		geometrical area	nominal area	nominal area		filtering surface area
Critical Reynolds number, $Re = VD/\nu$		rarely > 200 (cf. Section 1.4.1.3)	1,000–3,000	2,500		

Values of ξ, λ or K	turbulent	ordinary orifice 1·7–1·9 chamfered orifice 1 nozzle < 1 control valve 1–1·9	straight joint $\simeq$0·5 abrupt elbow $\simeq$1 banjo union 1·5–3 taps and valves 3–6	function of surface roughness: ~ 0·025		
	laminar	(seldom encountered) ξ increases slowly as Re decreases	ξ inversely proportional to Re	64/Re		function of the filter element (supplied by manufacturer)
Variations of ΔP with D, d or J for constant Q	turbulent	$1/D^4$		$1/D^5$		
	laminar	orifice in a very thin plate: $1/D^3$		$1/D^4$	$1/J^3$	$\simeq 1/d^4$
variations of ΔP with D, d or J for constant V	turbulent	0		$1/D$		
	laminar	orifice in a very thin plate: $1/D$		$1/D^2$	$1/J^2$	$\simeq 1/d^2$

* L, D, d, J, ϵ in cm, S in cm², ΔP in kg/cm², w in g/cm³, ν in cS (mm²/sec), V in m/sec.

† Where

$$Q = \frac{\pi}{128}\frac{1}{\nu\rho}\frac{D^4}{L}\,\Delta P$$

for laminar flow only.

‡ Where

$$Q = 2{\cdot}41\times 10^6\frac{1}{\nu w}\frac{D^4}{L}\,\Delta P$$

in laminar flow only.

If the reference areas for all the components are the same (connecting components)

$$(1/\sqrt{\xi})_{\text{tot}} = \Sigma\,(1/\sqrt{\xi})$$

ξ_{tot} being based on the common reference area.

Hence, for example, if there are two lines in parallel, the value of ξ is $\frac{1}{4}$ of that for a single line. If the components considered are restricting components with equal values of ξ, obviously they are equivalent to a single component of area $S_{\text{tot}} = \Sigma S$.

1.8. ORDER OF MAGNITUDE OF FLOW VELOCITIES

For parts of a circuit which join various components, such as the line joining a reservoir to a pump or a pump to an actuator, the coefficient of loss of head is given by

$$\xi_{\text{tot}} = \Sigma\,\xi + \Sigma\,\lambda L/D$$

and normally lies between 5 and 20.

The loss of head in these lines for a fluid of specific gravity 0·8 (kerosene or hydraulic fluid; cf. eqn. 3″) will lie between

$$\Delta P = 2\times 10^{-2} V^2 \quad \text{and} \quad \Delta P = 8\times 10^{-2} V^2$$

where ΔP is in kg/cm^2 and V is in m/sec; thus for

$$V = 10 \text{ m/sec}, \qquad \Delta P = 2\text{–}8 \text{ kg/cm}^2$$
$$V = 2 \text{ m/sec}, \qquad \Delta P = 0{\cdot}08\text{–}0{\cdot}32 \text{ kg/cm}^2.$$

This result shows that *in practice, the velocity of flow is limited to*

8–10 m/sec (300–400 in./sec) in high-pressure pipes (a loss of head of 2–8 kg/cm^2 is very small compared with the 100–150 kg/cm^2 produced by the fuel pump of a jet engine or the 200–300 kg/cm^2 produced by the pump in a hydraulic system);

2–3 m/sec (80–120 in./sec) in low-pressure pipes with a slightly pressurized reservoir;

1–2 m/sec (40–80 in./sec) in low-pressure circuits where the reservoir is left open to the atmosphere. In this case, as shown in Section 1.6.3, the loss of head may otherwise be greater than the pressure available at the reservoir.

APPENDIX 1.1

BERNOULLI'S EQUATION

Consider an elemental stream tube of cross-sectional area ds in the *steady flow of a perfect fluid* (*Figure 1.6*).

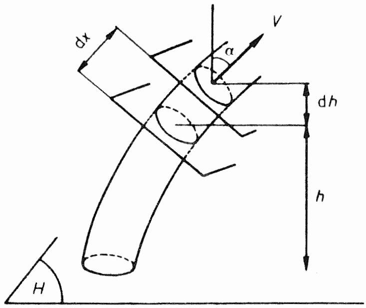


Figure 1.6

Let the velocity be V in an element of the stream tube length, dx, and
p = static pressure of the element
ρ = density
g = acceleration due to gravity
n = load factor (ng acceleration to which the fluid is submitted)
h = height of the element above reference plane H (horizontal if acceleration is that of gravity)
α = inclination of velocity vector, V, to a perpendicular from the reference plane, H.

Applying the equation of motion, $F = m(\mathrm{d}V)/(\mathrm{d}t)$ to the element and taking components in the x direction

$$-\varrho ng\,\mathrm{d}s\,\mathrm{d}x\cos\alpha - \frac{\mathrm{d}p}{\mathrm{d}x}\,\mathrm{d}x\,\mathrm{d}s = \varrho\,\mathrm{d}s\,\mathrm{d}x\,\frac{\mathrm{d}V}{\mathrm{d}t}$$

Putting dx cos α = dh and since $\mathrm{d}V/\mathrm{d}t = \mathrm{d}V/\mathrm{d}x\,.\,\mathrm{d}x/\mathrm{d}t = (\mathrm{d}V)/(\mathrm{d}x)V$, we have

$$\mathrm{d}p + \varrho ng\,\mathrm{d}h + \varrho V\,\mathrm{d}V = 0 \tag{1}$$

If the fluid is incompressible, ρ is constant, and the equation can be integrated directly to give

$$p + \rho ngh + \frac{\rho}{2}V^2 = \text{constant} = P_t$$

This is *Bernoulli's equation*. The constant is known as the *total pressure*. If for some reason, such as the loss of energy due to work done against friction, the total pressure decreases between two stations 1 and 2, we can write

$$\boxed{Pt_1 - Pt_2 = (p_1 - p_2) + \varrho ng\,(h_1 - h_2) + \frac{\varrho}{2}(V_1^2 - V_2^2) = \Delta P_{1-2}}$$

This is the *generalized form of Bernoulli's equation*.
The drop in total pressure between 1 and 2, ΔP_{1-2}, is called the *loss of head*.

APPENDIX 1.2

VISCOSITY

Absolute viscosity—Consider an infinitely small parallelepiped in a flow, defined by 2 planes, P_1 and P_2, perpendicular to the flow and distance dl apart, 2 planes, π_1 and π_2, perpendicular to the velocity gradient and distance dz apart, and 2 planes, Q_1 and Q_2, perpendicular to the planes defined and distance de apart (*Figure 1.7*).

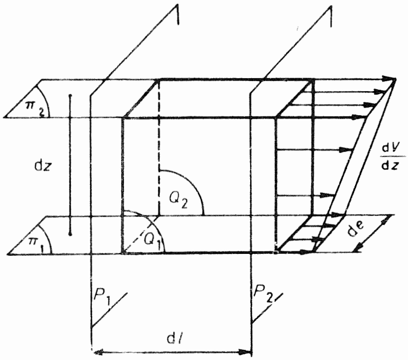


Figure 1.7

It can be shown experimentally that there is a relative retarding force between planes π_1 and π_2 in a direction parallel with the flow and is proportional to the area ds = dl . de, the velocity gradient, dV/dz, and to a coefficient characteristic of the fluid. This coefficient is called the *coefficient of viscosity*, μ, or sometimes just *the viscosity* and is defined by

$$\mathrm{d}F = \mu\, \mathrm{d}s \frac{\mathrm{d}V}{\mathrm{d}z}$$

i.e.

$$\mu = \frac{\mathrm{d}F}{\mathrm{d}s \dfrac{\mathrm{d}V}{\mathrm{d}z}}$$

Viscosity therefore has the dimensions

$$\frac{\mathrm{MLT^{-2}}}{\mathrm{L^3T^{-1}}} = \mathrm{ML^{-1}T^{-1}}$$

The C.G.S. unit of viscosity is the poise, Po (after Poiseuille):

$$1 \text{ poise} = 1 \text{ g/cm . sec}$$

Kinematic viscosity—More frequently used is the kinematic viscosity, ν, defined by

$$\nu = \mu/\rho$$

with the dimensions

$$\frac{ML^{-1}T^{-1}}{ML^{-3}} = L^2T^{-1}$$

The C.G.S. unit of kinematic viscosity is the stoke, S
In practice, kinematic viscosities are given in centistokes, cS

$$1\ S = 1\ cm^2/sec$$

$$1\ cS = 1\ mm^2/sec$$

It is sometimes necessary to know the kinematic viscosity in MTS units, i.e. in m^2/sec: *kinematic viscosity expressed in m^2/sec is equal to that in centistokes* $\times\ 10^{-6}$.

Chapter 11, Graph A gives values of the viscosity for different liquids at varying temperatures.

Note—In some books viscosity is still expressed in the old units, Redwood seconds in Britain, Saybolt seconds in America, degrees Engler and Barbey in France. These units are derived from reference values not directly related with the physical definition of viscosity. For example, the Engler degree is the ratio of the time taken for 200 cm^3 of the liquid considered to flow through an orifice of 2·8 cm diameter at 20°C to the time taken for 200 cm^3 of water to flow through the same orifice from the same reservoir. Saybolt seconds are the time taken for 60 cm^3 of the liquid to flow into a container through a certain orifice. It will be understood, therefore, that the relationship between these units, and between them and the more basic C.G.S. units given above, cannot be defined very rigorously. Graph B (Chapter 11) gives approximate conversion factors.

Viscosity of mixtures—It is usual to estimate the viscosity of a mixture of n liquids of viscosities $\nu_1, \nu_2, \ldots$ by

$$\nu = \nu_1 a_1 + \nu_2 a_2 + \ldots$$

where $a_1, a_2, \ldots$ are the volumetric proportions of each liquid in the mixture. Sometimes an empirical correction coefficient for $(a_1 - a_2)$ is added.

Some values of kinematic viscosity, in cS (mm^2/sec), *at t = 20°C*

Water	1·08
Water vapour	560
Air, at 1 atm	16
at an altitude of 20,000 m in the standard atmosphere	160
Alcohol	0·7
Sulphuric ether	0·3
Mercury	0·11
Petrol	0·6
Kerosene	1·7
Castor oil	1,000
Glycerine	650
Heavy medium oil	120
Hydraulic fluid	25

Note on Physical Interpretation of Reynolds Number—Consider an element of fluid similar to that shown in *Figure 1.7* where L is the reference length and V the reference velocity. The volume of the element is proportional to L^3, the kinetic energy to $K_1 = \rho L^3 V^2$, the viscous forces to $K_2 = \mu L V$ (see p. 26) and the work done by the viscous forces to $K_3 = \mu L^2 V$. Hence

$$\frac{K_1}{K_3} = \frac{\varrho L^3 V^2}{\mu L^2 V} = \frac{V L}{\nu} = Re$$

The Reynolds number thus indicates the ratio of the kinetic energy to the work done against viscous friction.

APPENDIX 1.3

LAMINAR FLOW IN PIPES

Consider an annulus in the laminar flow through a pipe bounded by two cylinders concentric with the pipe, radii r and $r + \mathrm{d}r$, and two cross-sectional planes, 0 and 1, perpendicular to the pipe (*Figure 1.8*), with

P_0, P_1 = pressure at sections 0 and 1
R, D = radius and diameter of the pipe
L = distance between the two planes
μ, ν = absolute viscosity and kinematic viscosity of the liquid
ρ, w = density and specific weight of the liquid
$V(r)$ = velocity of flow (a function of r); the positive direction is taken as the direction of flow, i.e. from 0 to 1.

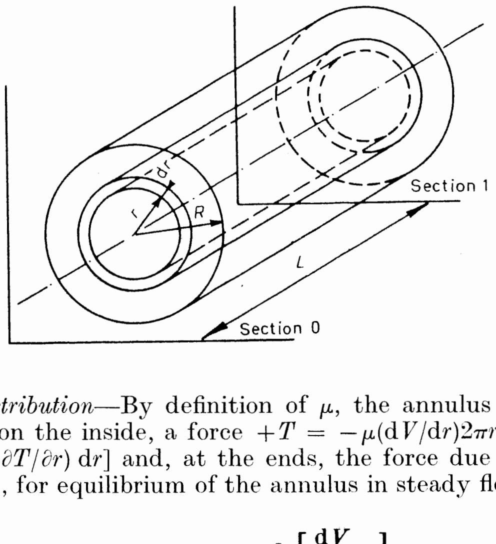


Figure 1.8

Determination of the velocity distribution—By definition of μ, the annulus of liquid considered is subject to, on the inside, a force $+T = -\mu(\mathrm{d}V/\mathrm{d}r)2\pi rL$; on the outside, a force $-[T + (\partial T/\partial r)\,\mathrm{d}r]$ and, at the ends, the force due to pressure $+(P_0 - P_1)2\pi r\,\mathrm{d}r$. Thus, for equilibrium of the annulus in steady flow

$$(P_0 - P_1)\,2\,\pi r\,\mathrm{d}r = +\frac{\partial T}{\partial r}\,\mathrm{d}r = -2\,\pi L\mu\,\frac{\partial\left[\dfrac{\mathrm{d}V}{\mathrm{d}r}\,r\right]}{\partial r}\,\mathrm{d}r$$

and, after integration

$$\frac{(P_0 - P_1)}{L\mu}\left[\frac{r^2}{2} + A\right] = -\frac{\mathrm{d}V}{\mathrm{d}r}\,r$$

which can be written

$$\frac{(P_0 - P_1)}{L\mu}\left[\frac{r}{2} + \frac{A}{r}\right]\mathrm{d}r = -\,\mathrm{d}V$$

Integrating again, we have

$$V = -\frac{(P_0 - P_1)}{L\mu}\left[\frac{r^2}{4} + A \operatorname{Log} r + B\right]$$

When $r = 0$, V is finite, and so $A = 0$.
When $r = R$, $V = 0$ (no velocity at the wall), and so

$$\boldsymbol{V = \frac{(P_0 - P_1)}{4\,L\mu}\,[R^2 - r^2]} \tag{1}$$

The velocity distribution for laminar flow in a pipe is parabolic.

Determination of the flow—The volume flow is given by

$$Q = \int_0^R V\,\mathrm{d}S = \int_0^R V\,2\,\pi r\,\mathrm{d}r$$

Thus

$$Q = \int_0^R \frac{(P_0 - P_1)}{4\,L\mu}\,2\,\pi\,[R^2 r\,\mathrm{d}r - r^3\,\mathrm{d}r]$$

$$= \frac{\pi}{8}\,\frac{(P_0 - P_1)}{L\mu}\,R^4$$

This expression is usually given as a function of the diameter:

$$\boldsymbol{Q = \frac{\pi}{128}\,\frac{1}{\mu}\,\frac{D^4}{L}\,(P_0 - P_1)} \tag{2}$$

If we write $P_0 - P_1 = \Delta P$ and use the kinematic viscosity, $\nu = \mu/\rho$

$$\boldsymbol{Q = \frac{\pi}{128}\,\frac{1}{\nu\rho}\,\frac{D^4}{L}\,\Delta P} \tag{3}$$

$$\boldsymbol{\Delta P = \frac{128}{\pi}\,\nu\rho\,\frac{L}{D^4}\,Q} \tag{4}$$

and finally, with the units
L and D in cm
Q in cm^3/sec
ν in cS (mm^2/sec)
ΔP in kg/cm^2
w the specific weight in g/cm^3 (the same numerical value as specific gravity):

$$Q = 2{\cdot}41 \times 10^6\,\frac{1}{\nu w}\,\frac{D^4}{L}\,\Delta P \tag{5}$$

$$\Delta P = 0{\cdot}415 \times 10^{-6}\,\nu w\,\frac{L}{D^4}\,Q \tag{6}$$

Determination of the coefficient λ—The coefficient λ is defined by the equation

$$\Delta P = \frac{\varrho}{2}\,\lambda\,\frac{L}{D}\,\frac{Q^2}{S^2}$$

We can determine λ by equating this expression with eqn. (4). After simplification, and putting $Q/S = V$ and $S = \pi D^2/4$

$$\lambda = \frac{64\ \nu}{V \cdot D}$$

or, in its more usual form (the required expression has to be of the form C/Re, cf. p. 28)

$$\boxed{\lambda = \frac{64}{Re}} \tag{7}$$

APPENDIX 1.4

LAMINAR FLOW BETWEEN PARALLEL PLATES

Consider a thin elemental lamina, in the laminar flow between two parallel plates, bounded by two planes perpendicular to the flow, 0 and 1; two planes parallel to the walls, whose ordinates are y and $y+\mathrm{d}y$, and two planes, A and B, perpendicular to the latter (*Figure 1.9*), with

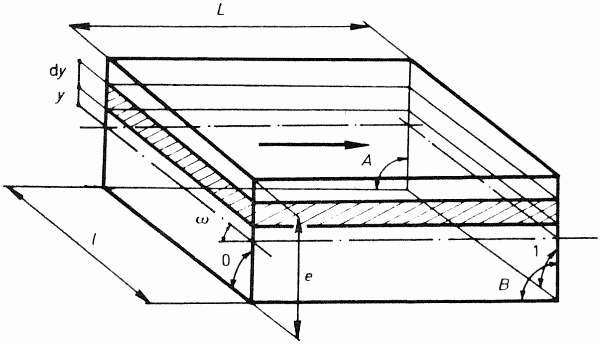


Figure 1.9

P_0, P_1 = the pressures at planes 0 and 1
e = distance between the two plates
L = distance between planes 0 and 1
l = distance between planes A and B
μ, ν = absolute viscosity and kinematic viscosity of the liquid
ρ, w = density and specific weight of the liquid
$V(y)$ = flow velocity (a function of y).

The positive direction is taken as the direction of flow and the origin of the y ordinates is the plane w, equidistant from the two plates.

Determination of the velocity distribution—The forces acting on the elemental lamina are

$$+\ T = -\mu L l \frac{\mathrm{d}V}{\mathrm{d}y}$$

$$-\left(T + \frac{\partial T}{\partial y}\,\mathrm{d}y\right)$$

$$(P_0 - P_1)\, l\, \mathrm{d}y$$

Thus the equation of equilibrium under conditions of steady flow is

$$(P_0 - P_1)\, l\, \mathrm{d}y = +\frac{\partial T}{\partial y}\,\mathrm{d}y = -\mu L l \frac{\mathrm{d}^2 V}{\mathrm{d}y^2}\,\mathrm{d}y$$

Integrating twice, we have

$$V = -\frac{(P_0 - P_1)}{L\mu}\left[\frac{y^2}{2} + Ay + B\right]$$

Since the flow is symmetrical with respect to the plane ω, $A = 0$, and since the velocity is zero at the walls (ordinates $\pm y_0$)

$$V = \frac{P_0 - P_1}{2L\mu}(y_0^2 - y^2) \tag{1}$$

The velocity distribution for laminar flow between parallel plates is parabolic (cf. p. 30).

Determination of the flow—The volume flow is given by

$$Q = 2\int_0^{y_0} lV(y)\,dy$$

$$Q = \frac{(P_0 - P_1)\,l}{L\mu}\int_0^{y_0}(y_0^2 - y^2)\,dy$$

$$= \frac{2}{3}\,\frac{(P_0 - P_1)\,l y_0^3}{L\mu}$$

or, by putting $e = 2y_0$

$$Q = \frac{1}{12}\,\frac{1}{\mu}\,\frac{le^3}{L}(P_0 - P_1) \tag{2}$$

Writing $P_0 - P_1 = \Delta P$ and $\nu = \mu/\rho$

$$Q = \frac{1}{12}\,\frac{1}{\nu\rho}\,\frac{le^3}{L}\,\Delta P \tag{3}$$

$$\Delta P = 12\,\nu\rho\,\frac{L}{e^3 l}\,Q \tag{4}$$

and finally, with the usual C.G.S. units

L, l and e in cm
Q in cm^3/sec
ν in cS (mm^2/sec)
ΔP in kg/cm^2
w, the specific weight, in g/cm^3 (same numerical value as the specific gravity):

$$Q = 8{\cdot}18 \times 10^6\,\frac{1}{\nu w}\,\frac{le^3}{L}\,\Delta P \tag{5}$$

$$\Delta P = 0{\cdot}122 \times 10^{-6}\,\nu w\,\frac{L}{le^3}\,Q \tag{6}$$

Determination of the coefficient λ—If, as in Appendix 1.3, we equate eqn. (4) with the basic relationship defining λ

$$\Delta P = \frac{\varrho}{2}\,\lambda\,\frac{L}{e}\,\frac{Q^2}{S^2}$$

we have, after simplification and putting $Q/S = V$ and $S = le$

$$\lambda = \frac{24\,\nu}{Ve}$$

or, in its more usual form

$$\boxed{\lambda = \frac{24}{Re}} \tag{7}$$

where Re is the Reynolds number based on the distance e between the two plates.

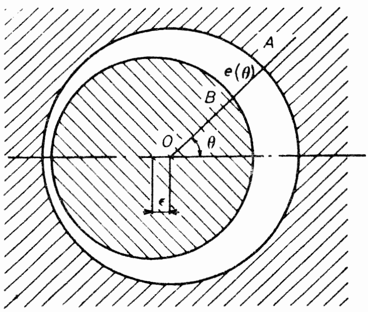
APPENDIX 1.5

LAMINAR FLOW BETWEEN TWO CYLINDERS' CONCENTRIC OR ECCENTRIC

Let (*Figure 1.10*)

L = length of flow considered
D_0, D_i, D = diameters of outer and inner cylinders and mean diameter
R_0, R_i, R = the corresponding radii
$J = D_0 - D_i$ = diametral clearance
$j = R_0 - R_i$ = radial clearance
ϵ = eccentricity (distance between the two axes)
θ = angle between a radius R_0 and the plane containing the axes of the two cylinders
$e(\theta)$ = clearance at any given value of θ (i.e. the distance AB).

Figure 1.10



(1) *Concentric cylinders*—Applying directly the equations established in Appendix 1.4 and putting $l = \pi D$ and $e = j$

$$Q = \frac{\pi}{12} \frac{1}{\nu\varrho} \frac{Dj^3}{L} \Delta P \tag{1}$$

or, in terms of the diametral clearance, J, which is more generally used in mechanics

$$Q = \frac{\pi}{96} \frac{1}{\nu\rho} \frac{DJ^3}{L} \Delta P \tag{2}$$

With

L, D and J in cm
Q in cm^3/sec
ν in cS (mm^2/sec)
ΔP in kg/cm^2
w, the specific weight, in g/cm^3

$$Q = 3{\cdot}21 \times 10^6 \frac{1}{\nu w} \frac{DJ^3}{L} \Delta P \tag{3}$$

$$\Delta P = 0{\cdot}312 \times 10^{-6} \nu w \frac{L}{DJ^3} Q \tag{4}$$

(2) *Eccentric cylinders*

Determination of $e(\theta)$—ϵ being very small compared with R,

$$OB = x \triangleq R_i - \epsilon \cos \theta$$

$$e = R_0 - x = j + \epsilon \cos \theta$$

Determination of Q—Consider an elemental strip in the flow, which subtends an angle $d\theta$ to the centre, and apply the result from Appendix 1.4

$$Q = \frac{1}{12} \frac{1}{\nu \varrho} \frac{l e^3}{L} \Delta P$$

which gives

$$dQ = \frac{1}{12} \frac{1}{\nu \varrho} \frac{e^3}{L} R_0 \, d\theta \, \Delta P = \frac{D_0 \Delta P}{24 \nu \varrho L} (j + \varepsilon \cos \theta)^3 \, d\theta$$

$$Q = \int_0^{2\pi} dQ = \frac{D_0 \Delta P}{24 \nu \varrho L} j^3 \int_0^{2\pi} \left(1 + \frac{\varepsilon \cos \theta}{j}\right)^3 d\theta$$

$$= \frac{\pi}{12} \frac{1}{\nu \varrho} \frac{D_0 j^3}{L} \left(1 + \frac{3}{2} \frac{\varepsilon^2}{j^2}\right) \Delta P \tag{5}$$

With $J = 2j$ and since $D_0 \triangleq D$:

$$\mathbf{Q = \frac{\pi}{96} \frac{1}{\nu \rho} \frac{DJ^3}{L} \left(1 + \frac{3}{2} \frac{\epsilon^2}{(J/2)^2}\right) \Delta P} \tag{6}$$

Eccentricity of the two cylinders increases the flow. When the eccentricity is a maximum ($\epsilon_{max} = J/2$), *the flow is* $2\frac{1}{2}$ *times greater than for the concentric case.*

Note—It is interesting to compare the flow produced in two ducts of the same length and with the same cross-sectional area, one being cylindrical and the other the annular space between two cylinders.

These flows have been dealt with in Appendices 1.3 and 1.5; expressed in terms of the common cross-sectional area, S, they are

$$Q_1 = \frac{\pi}{8} \frac{1}{\mu} \frac{R^4}{L} \Delta P = \frac{1}{8\pi} \frac{1}{\mu} \frac{S^2}{L} \Delta P$$

and

$$Q_2 = \frac{\pi}{12} \frac{1}{\mu} \frac{D j^3}{L} \Delta P \, a = \frac{1}{12\pi} \frac{1}{\mu} \frac{S^2}{L} \frac{j}{D} \Delta P \, a$$

where α is the coefficient of eccentricity $\alpha = 1 + \frac{3}{2}\epsilon^2/j^2$, i.e.

$$\frac{Q_2}{Q_1} = \frac{2}{3} \frac{j}{D} a$$

Since $1 \leqslant \alpha \leqslant 2\cdot5$

$$\frac{2}{3}\frac{j}{D} \leqslant \frac{Q_2}{Q_1} \leqslant \frac{5}{3}\frac{j}{D}$$

As j is much smaller than D, the flow in an annular duct is much less than that in a pipe of the same cross-sectional area.

APPENDIX 1.6

LOSS OF HEAD AT AN ABRUPT CHANGE OF CROSS-SECTIONAL AREA

(1) *Enlargements*—For a pipe of cross-sectional area S, increased very abruptly to S_1 (*Figure 1.11*), we define a coefficient of expansion by $\alpha = S_1/S$. Consider two sections across the larger pipe. One, S_1, placed at a distance of 8 or 10 D_1 downstream of the expansion, i.e. far enough away for the flow to have settled

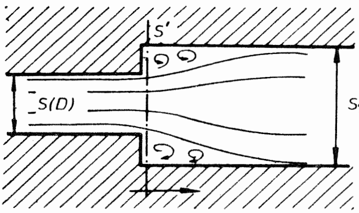


Figure 1.11

down so that the pressure, p, is constant over the cross-section, and the other, S', immediately downstream of the expansion. In the latter, the stream lines are still parallel with the axis of the pipe, and the pressure, p', can also be considered constant over the cross-section. There is no change in pressure between the flow at the centre and the eddies in the corners (Gibson's hypothesis).

Applying the momentum equation to the fluid between S' and S_1

$$p_1 S_1 - p' S_1 + m(V_1 - V') = 0$$

where m is the mass flow. But

$$S_1 = \frac{Q}{V_1} = \frac{m}{\varrho V_1}$$

so we have

$$p_1 - p' = -\varrho V_1 (V_1 - V')$$

For a perfect fluid, where there are no losses, the pressure p_{1B} at section S_1 can be found from Bernoulli's equation

$$p_{1B} - p' = -\frac{\varrho}{2}(V_1^2 - V'^2)$$

Thus, the loss of head is

$$\Delta P = p_{1B} - p_1 = (p_{1B} - p') - (p_1 - p') = \frac{\varrho}{2}[V'^2 - V_1^2 - 2V(V' - V_1)]$$

$$\Delta P = \frac{\varrho}{2}(V' - V_1)^2$$

and the coefficient ξ based on the area S is given by

$$\xi = \frac{\Delta P}{\varrho V^2/2} = \left(\frac{V'}{V} - \frac{V_1}{V}\right)^2$$

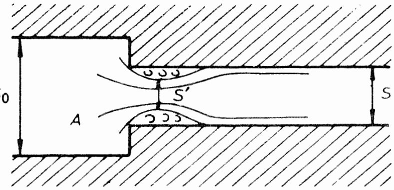
But $V' = V$ and $V_1/V = 1/\alpha$, and so

$$\xi = \left(1 - \frac{1}{\alpha}\right)^2$$

This equation is confirmed by practical results for turbulent flow.

(2) *Reduction of area* (*Figure 1.12*)—If the reduction of area is abrupt, the stream of liquid contracts so that, at a point downstream of the reduction (at the *vena contracta*), its cross-sectional area has a minimum value, $S' = \beta S$. The coefficient β, known as the coefficient of contraction, depends upon $\gamma = S_0/S$, the coefficient of reduction, and also on the shape of the edge at A.

Figure 1.12



Assume that β is known and that there are no losses between S_0 and S'. Applying the momentum equation between S' and S and assuming, as before, that the pressure is constant across these sections

$$pS - p'S + m(V - V') = 0$$

and, since

$$S = \frac{Q}{V} = \frac{m}{\varrho V},$$

$$p - p' = -\varrho V(V - V')$$

Now, for a perfect fluid

$$p_B - p' = -\frac{\varrho}{2}(V^2 - V'^2)$$

and the loss of head is therefore

$$\Delta P = p_B - p = (p_B - p') - (p - p') = \frac{\varrho}{2}[V'^2 - V^2 - 2V(V' - V)]$$

$$\Delta P = \frac{\varrho}{2}(V' - V)^2$$

The coefficient ξ, based on the area S, is given by

$$\xi = \frac{\Delta P}{\varrho V^2/2} = \left(\frac{V'}{V} - 1\right)^2$$

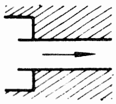
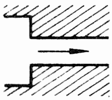
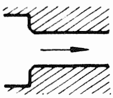
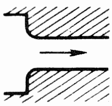
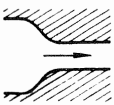
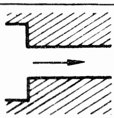
and, since

$$\frac{V'}{V} = \frac{S}{S'} = \frac{1}{\beta},$$

$$\xi = \left(\frac{1}{\beta} - 1\right)^2$$

The problem is not completely solved, since the value of β was assumed. Values of β are given in *Table 1.1*.

Table 1.1 (from Th. Oniga[1])

			β	ξ
$\gamma = S_0/S > 10$		Protruding edge (Borda mouthpiece)	0·47–0·50	1·3–1
		Square edge	0·61–0·65	0·40–0·30
		Bevelled edge	0·7–0·8	0·20–0·06
		Rounded edge	0·9	0·012
		Streamlined entrance	0·99	negligible
Any value of γ		Square edge	$0{\cdot}63+\dfrac{0{\cdot}37}{\gamma}$	$\dfrac{\gamma-1}{1{\cdot}7\gamma+1}$

(3) *Local change of cross-sectional area in a pipe*—The loss of head is easily deduced from the preceding analysis.

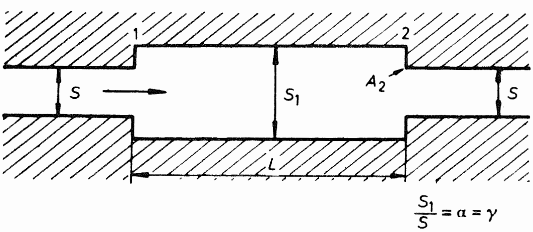


Figure 1.13

(*a*) Consider an abrupt increase in cross-sectional area for a length L and base the coefficient ξ on the area of the *pipe* (*Figure 1.13*). If L is about 8 or 10 D (or, in practice, even as low as 5 or 6 D):

$$\left.\begin{aligned} \xi_1 &= \left(1 - \frac{1}{a}\right)^2 \\ \xi_2 &= \left(\frac{1}{\beta\,(a)} - 1\right)^2 \end{aligned}\right\} \quad \xi = \xi_1 + \xi_2$$

If α is large,

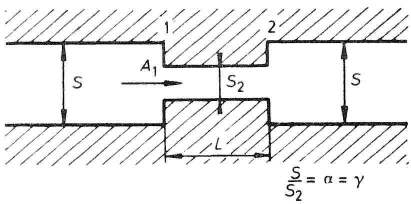
$$\xi = 1 + \xi_2$$

ξ_2 depends on the shape of the edge at A_2 (*Table 1.1*).

(*b*) Consider an abrupt reduction in the cross-sectional area for a length L and base the coefficient ξ on the smaller area, i.e. S_2 (*Figure 1.14*):

$$\left.\begin{aligned} \xi_1 &= \left(\frac{1}{\beta\,(a)} - 1\right)^2 \\ \xi_2 &= \left(1 - \frac{1}{a}\right)^2 \end{aligned}\right\} \quad \xi = \xi_1 + \xi_2$$

Figure 1.14



If α is large,

$$\xi = 1 + \xi_1$$

ξ_1 depends on the edge at A_1 (*Table 1.1*).

In order to determine the coefficients ξ', based on the cross-sectional area S, the coefficients ξ should be multiplied by α^2.

REFERENCES

[1] Oniga, Th. *Calcul des Tuyaux*, Paris (Matemine) 1949

2

ACCUMULATION, TRANSMISSION AND DISSIPATION OF HYDRAULIC ENERGY

2.1. THE SUPPLY OF A COMPRESSED FLUID

2.1.1. GENERAL CONSIDERATIONS

In order to provide a compressed fluid, a certain amount of energy is required which consists of the sum of work done in compressing the fluid, W_1, and of that done in discharging the fluid at constant pressure, W_2. We shall examine these two types of work and determine their theoretical values (assuming 100 per cent efficiency).

To do this, consider the compressor as equivalent to a piston moving in a closed cylinder of cross-sectional area S, provided with two valves R_0 and R_1 which connect it with an external supply of fluid at constant pressure P_0, and an accumulator, C_1, filled with fluid at constant pressure, P_1 (*Figure 2.1*).

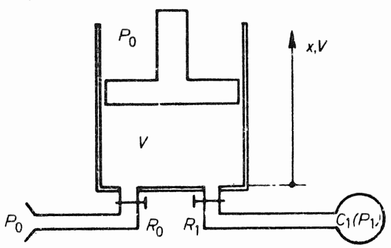


Figure 2.1

The piston is initially at $x = x_0$, with the valve R_0 open and R_1 closed. Then valve R_0 is closed, leaving a mass, M, of fluid at volume V_0 and pressure P_0 in the cylinder. Next, the piston is moved down the cylinder to a position $x = x_1$, thus compressing the fluid to a volume V_1 and pressure P_1. To carry out this operation, it is necessary to do work, W_1, on the piston, where

$$W_1 = -\int_{x_0}^{x_1} (P - P_0)\, S\, \mathrm{d}x = -\int_{V_0}^{V_1} (P - P_0)\, \mathrm{d}V \tag{1}$$

In the P–V diagram, W_1 is represented by the curved triangle ABE (*Figure 2.2*).

If valve R_1 is now opened and the piston moved right down to the bottom of the cylinder, the mass M of fluid is pushed into the storage vessel, C_1. The work done on the piston is

$$W_2 = -\int_{x_1}^{0} (P_1 - P_0)\, S\, \mathrm{d}x = -\int_{V_1}^{0} (P_1 - P_0)\, \mathrm{d}V = (P_1 - P_0)\, V_1 \quad (2)$$

In the P–V diagram, W_2 is represented by the rectangle $BCDE$ (*Figure 2.2*).

Finally, R_1 is closed, R_0 opened and the piston returned to its initial position, without any further work being done.

W is therefore equal to $W_1 + W_2$ and is represented by the area $ABCDE$ (*Figure 2.2*).

The work done in compression, W_1, is stored in the fluid, while the work W_2 is used only to move the fluid from the piston to the accumulator.

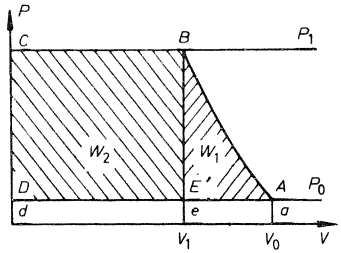


Figure 2.2

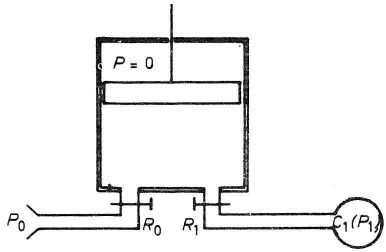


Figure 2.3

Note—The type of compressor generally considered in thermodynamics takes a slightly different form. It has a cylinder closed at both ends, with a vacuum in the upper chamber (*Figure 2.3*).

Under these conditions, and referring to the P–V diagram in *Figure 2.2*, work done in compression (area $AB\ ea$):

$$W_1' = -\int_{V_0}^{V_1} P\, \mathrm{d}V \; (= JC_v\, \Delta T) \quad (3)$$

Work done in discharging fluid (area $BC\ de$):

$$W_2' = -\int_{V_1}^{0} P_1\, \mathrm{d}V = P_1 V_1 \quad (4)$$

Work done during the induction stroke (area $AadD$):

$$W_3' = -P_0 V_0$$

Total work done (area $ABCDE$):

$$W' = W_1' + W_2' + W_3' = W\, (= JC_p\, \Delta T)$$

Even if the fluid absorbs the work W_1' efficiently and then expands to surroundings at pressure P_0 in a machine which converts all the pressure energy to work, it will only do work W_1. The surroundings will absorb the remaining energy:

$$W_1' - W_1 = (V_0 - V_1)\, P_0$$

For this reason we shall consider the other type of compressor.

2.1.2. HYDRAULIC LIQUIDS

The bulk modulus, B, of liquids is defined by the equation (see Section 5.3.2)

$$\frac{\mathrm{d}V}{V} = -\frac{\mathrm{d}P}{B}$$

B has the dimensions of pressure and has a high numerical value, much larger than the pressure generally used in hydraulic systems.

This means that $\mathrm{d}V$ is small compared with V, and the defining equation can be written

$$\frac{\mathrm{d}V}{V_m} = -\frac{\mathrm{d}P}{B}$$

where V_m is the mean volume, $(V_0 + V_1)/2$. Thus, in *Figure 2.2*, the curve AB can be replaced by a straight line AB' whose gradient is that of the curve AB at the point $V = V_m$, i.e. $\mathrm{d}P/\mathrm{d}V = -B/V_m$ (*Figure 2.4*).

Either by substituting $\mathrm{d}V = -(V_m/B)\,\mathrm{d}P$ in the expressions for W_1 and W_1' or by directly evaluating the areas in *Figure 2.4*, it can be shown that

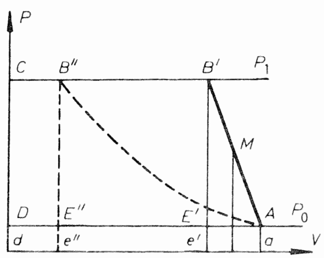


Figure 2.4

$$W_1 = \frac{V_m}{B}\,\frac{(P_1 - P_0)^2}{2} \tag{5}$$

$$W_2 = V_1\,(P_1 - P_0) \tag{6}$$

$$W_1' = \frac{V_m}{B}\,\frac{(P_1^2 - P_0^2)}{2} \tag{7}$$

$$W_2' = V_1 P_1 \tag{8}$$

$$W_3' = -\,V_0 P_0 \tag{9}$$

and

$$W = W_1 + W_2 = W_1' + W_2' + W_3'$$

$$= \left[V_1 + \frac{V_m}{2B}(P_1 - P_0)\right](P_1 - P_0) = \left[V_1 + \frac{V_0 - V_1}{2}\right](P_1 - P_0)$$

$$\boxed{W = V_m\,(P_1 - P_0)} \tag{10}$$

For the liquids used in hydraulics, B is of the order of 15,000–20,000 kg/cm^2 (200,000–300,000 lb./in.^{2}), and the maximum pressures in actual installations never exceed 300 kg/cm^2 (4,250 lb./in.^{2}). Thus

$$\frac{W_1}{W} = \frac{P_1 - P_0}{2B} \leqslant \frac{1}{100}$$

In the supply of a hydraulic liquid under pressure, the work done in compression is negligible compared with that done in discharging the fluid.

2.1.3. GASES

For a gas, if γ is the ratio of the specific heats C_p/C_v, we have

(*a*) *For an adiabatic process* ($PV^\gamma = P_0 V_0^\gamma$):

$$W_1 = -\int_{V_0}^{V_1} (P-P_0)\,\mathrm{d}V = P_0 V_0 \left\{\frac{1}{\gamma-1}\left[\left(\frac{P_1}{P_0}\right)^{(\gamma-1)/\gamma} - 1\right] + \left(\frac{P_1}{P_0}\right)^{-1/\gamma} - 1\right\} \qquad \text{(area } AB''E'')$$

$$W_2 = (P_1-P_0)V_1 = P_0 V_0 \left\{\left(\frac{P_1}{P_0}\right)^{(\gamma-1)/\gamma} - \left(\frac{P_1}{P_0}\right)^{-1/\gamma}\right\} \qquad \text{(area } B''CDE'')$$

$$W'_1 = -\int_{V_0}^{V_1} P\,\mathrm{d}V = P_0 V_0 \frac{1}{\gamma-1}\left[\left(\frac{P_1}{P_0}\right)^{(\gamma-1)/\gamma} - 1\right] \qquad \text{(area } aAB''e'')$$

$$W'_2 = P_1 V_1 = P_0 V_0 \left(\frac{P_1}{P_0}\right)^{(\gamma-1)/\gamma} \qquad \text{(area } B''Cde)$$

$$W'_3 = -P_0 V_0 \qquad \text{(area } aADd)$$

$$W = W_1 + W_2 = W'_1 + W'_2 + W'_3$$

$$= P_0 V_0 \frac{\gamma}{\gamma-1}\left[\left(\frac{P_1}{P_0}\right)^{(\gamma-1)/\gamma} - 1\right] \qquad \text{(area } AB''CD)$$

When the pressure ratio, P_1/P_0, is high, $W'_1 \to W_1$, $W'_2 \to W_2$ and $W'_3/W \to 0$. Therefore

$$\frac{W_1}{W} \to \frac{W'_1}{W} = \frac{1}{\gamma}$$

For air, ($\gamma = 1{\cdot}4$), $W_1/W \to 0{\cdot}71$.

(*b*) *For an isothermal process* ($PV = P_0 V_0$):

$$W_1 = P_0 V_0 \left[\frac{P_0}{P_1} - 1 + \log\frac{P_1}{P_0}\right]$$

$$W_2 = P_0 V_0 \left[1 - \frac{P_0}{P_1}\right]$$

$$W'_1 = P_0 V_0 \log\frac{P_1}{P_0}$$

$$W'_2 = P_0 V_0$$

$$W'_3 = P_0 V_0{}^*$$

$$W = P_0 V_0 \log\frac{P_1}{P_0}$$

* The equality of W'_2 and $-W'_3$ on the one hand, and of W'_1 and W on the other, is obvious since the PV curve is a symmetrical hyperbola (PV = constant).

When P_1/P_0 increases, W_1/W tends slowly to 1. For gases, then, the work done in discharging the fluid becomes negligible compared with the work done in compression.

Energy can be stored in gases but not in liquids.

Accumulators (energy storage components), wrongly called hydraulic accumulators, are really pneumatic accumulators since the liquid itself merely transmits the energy*.

2.2. HYDRAULIC POWER

2.2.1. POWER CARRIED BY A LIQUID UNDER PRESSURE

Dividing both sides of eqn. (10) by unit time, we have

$$H = Q_m (P_1 - P_0)$$

where H is the power required and Q_m the mean volume flow, i.e. calculated from the density ρ_m, the mean of ρ_0 under $P = P_0$ and ρ_1 under $P = P_1$.

Small variations in density are often ignored, and we write

$$H = Q \, \Delta P \tag{11}$$

or, with the units generally used in hydraulics (see Chapter 11)

$$H_{(\mathrm{kgm/s})} = 10 \, Q_{(\mathrm{l/s})} \Delta P_{(\mathrm{kg/cm^2})} \tag{12}$$

or

$$H_{(\mathrm{kW})} = \frac{1}{10{,}2} \, Q_{(\mathrm{l/s})} \, \Delta P_{(\mathrm{kg/cm^2})} \tag{12'}$$

(Note that the power does not depend on the density. We could have anticipated this, since there is no acceleration of the fluid. Thus, in theory at any rate, there is no disadvantage in the use of light fluids in order to keep the weight of an installation to a minimum.)

2.2.2. POWER AND TORQUE ABSORBED BY A PUMP OR PROVIDED BY A HYDRAULIC MOTOR

Let

v = swept volume (i.e. displaced by 1 revolution) of pump or motor
ω = angular speed, in rad/sec (n in rev/sec)
T = torque on the shaft.

For the pump, we define

$$\text{volumetric efficiency } \eta_v = \left(\frac{\text{actual volume flow from pump}}{\text{theoretical flow, } nv}\right)$$

$$\text{overall efficiency, } \eta_0 = \left(\frac{\text{hydraulic power output from pump}}{\text{mechanical power input to pump}}\right)$$

* See Chapter 10, Example 2.

For the motor

$$\text{volumetric efficiency, } \eta_v' = \left(\frac{\text{theoretical flow, } nv}{\text{actual flow to the motor}}\right)$$

$$\text{overall efficiency, } \eta_0' = \left(\frac{\text{mechanical power output from motor}}{\text{hydraulic power input to motor}}\right)$$

The following equations will apply:

	Pump	*Motor*
Shaft power	$H = \frac{1}{\eta_0} Q\,\Delta P$	$H = \eta_0' Q\,\Delta P$
	$H_{(\text{kg.m/sec})} = \frac{10}{\eta_0} Q_{(\text{l./sec})} \Delta P_{(\text{kg/cm}^2)}$	$H_{(\text{kg.m/sec})} = 10\eta_0' Q_{(\text{l./sec})} \Delta P_{(\text{kg/cm}^2)}$
	$H_{(\text{kW})} = \frac{1}{10{\cdot}2\eta_0} Q_{(\text{l./sec})} \Delta P_{(\text{kg/cm}^2)}$	$H_{(\text{kW})} = \frac{\eta_0'}{10{\cdot}2} Q_{(\text{l./sec})} \Delta P_{(\text{kg/cm}^2)}$
Rotational speed	$\omega = \frac{2\pi}{\eta_v} \frac{Q}{v}$	$\omega = 2\pi\eta_v' \frac{Q}{v}$
	$n_{(\text{rev/sec})} = \frac{1}{\eta_v} \frac{Q_{(\text{l./sec})}}{v_{(\text{l.})}}$	$n_{(\text{rev/sec})} = \eta_v' \frac{Q_{(\text{l./sec})}}{v_{(\text{l.})}}$
Shaft torque	$T = \frac{H}{\omega} = \frac{\eta_v}{2\pi\eta_0} v\,\Delta P$	$T = \frac{H}{\omega} = \frac{\eta_0'}{2\pi\eta_v'} v\,\Delta P$
	$T_{(\text{m.kg})} = \frac{10}{2\pi} \frac{\eta_v}{\eta_0} v_{(\text{l.})} \Delta P_{(\text{kg/cm}^2)}$	$T_{(\text{m.kg})} = \frac{10}{2\pi} \frac{\eta_0'}{\eta_v'} v_{(\text{l.})} \Delta P_{(\text{kg/cm}^2)}$

2.3. STORAGE OF ENERGY FOR HYDRAULIC SYSTEMS

In self-contained assemblies (aircraft, engines), the problem of storing energy for the hydraulic circuit arises. The following are generally used: accumulators with air or an inert gas; electric batteries (these require an electric motor which, in turn, needs a hydraulic pump, a reservoir and in most cases a small buffer accumulator); a source of chemical energy: solid fuel, hydrogen peroxide, kerosene, liquefied gas, etc. with their associated equipment; a diversion of some of the propulsive energy from the engine, e.g. of mechanical power to a pump; bleeding air from the compressor of a jet engine to drive a turbopump, etc. The choice is mainly determined by the allowable weight of the assembly.

If the total energy to be stored is E and the maximum power H, the weight of the complete generating assembly can be approximately expressed in the form

$$W = \alpha E + \beta H$$

Table 2.1. Storage of energy

1	2	3	4	5	6	7
Type of energy	*Type of accumulator*	*Energy per unit weight*, kg, m/kg				*Definition of parameters; remarks* (in brackets: numerical values used for calculation of columns 4 and 6)
		Stored energy		*Hydraulic energy recovered*		
		Algebraic expression	*Numerical value* (see col. 7)	*Algebraic expression*	*Numerical value* (see col. 7)	
Mechanical deformation energy	compressed liquid	$\frac{10P_1^2}{2Bw}$	26	not used		P_1 = pressure, in kg/cm² (300) B = bulk modulus in kg/cm² (20,000) w = specific weight (0·86 g/cm³)
	steel in tension or compression	$\frac{1{,}000t_1^2}{2Ew} = \frac{t_1^2}{328}$	30	not used		t_1 = stress, in kg/mm² (100) E = Young's modulus, in kg/mm² (21,000) w = specific weight (7·8 g/cm³)
	coiled spring accumulator (steel)	$\frac{1{,}000t_1^2}{4Gw} = \frac{t_1^2}{250}$	40	$\frac{t_1^2-t_2^2}{250}$	22	t_1 = stress, in kg/mm² (100) G = shear modulus, in kg/mm² (8,000) w = specific weight (7·8 g/cm³) Col. 5 and 6: spring operating between t_1 (100) and t_2 (66·7)
	rubber spring accumulator	—	~400	—	~200	largely depends on nature of rubber, type of spring and mode of use
	compressed air	$JC_vT_1 = \frac{P_1V_1}{\gamma-1}T_1 = 72T_1$	20,700			adiabatic expansion to zero pressure (not to P_1) T_1 = absolute initial temperature, in °K (288 = 15°C)
	oil and air accumulator (cf. Chapter 10, Example 2)	$\frac{72T_1}{1+K_1+K_2}$	2,500	$\frac{72T_1}{1+K_1+K_2}\times\left[1-\left(\frac{V_1}{V_2}\right)^{0.4}\right]$	375	adiabatic expansion; volumes V_1, V_2; $V_1/V_2 = \frac{2}{3}$ (volume of oil used $= V_2 - V_1$)
				$\frac{6740T_1}{1+K_1+K_2}\log\frac{V_2}{V_1}$	410	isothermal expansion; T_1 = initial absolute temperature, in °K (288 = 15°C) K_1 = weight of container necessary for air only, per kg of air (4·1)* K_2 = weight of oil and container (per kg of air) (3·2)*

Mechanical deformation energy	air cylinder +pneumatic motor + hydraulic pump (cf. Chapter 10, Example 2)	$\frac{72T_1}{1+K_1}$	4,050	$\frac{100T_1}{1+K_1}\times\left[1-\left(\frac{P_2}{P_1}\right)^{0.287}\right]\eta_m.\eta_P$	1.090‡	pre-expansion without any work being done from P_1 (300 kg/cm²) to P_1' (7 kg/cm²) by means of a pressure-reducing valve adiabatic expansion from P_1 to P_2 (1 kg/cm²) in a motor of efficiency η_m (0·6)† efficiency of hydraulic pump, η_P (0·75) (equation valid only if $P_1' < P_1$ (see Section 8.2)
Kinetic	flywheel +hydraulic pump	$\frac{V_t^2}{2g}$	4,500	$\frac{V_{t_1}^2-V_{t_2}^2}{2g}\eta_P$	2,500‡	V_t = tangential speed of flywheel, in m/sec V_{t_1} = initial speed (300): V_{t_2} = final speed (150) efficiency of hydraulic pump η_P (0·75)
Electrical	battery + motor and pump	E_e	~8,000	$E_e\eta_m\eta_P$	~4,000‡§	lead battery; efficiency of electric motor, η_m
			~18,000		~9,000‡§	nickel–iron battery; efficiency of pump, η_P
			~30,000		1,500‡§	zinc–silver oxide battery; ($\eta_m\eta_P$ = 0·5)
Chemical	hydrogen peroxide +motor and pump	E_c	160,000	$E_c\eta_t\eta_P$	40,000‡	pure hydrogen peroxide; catalytic decomposition; efficiency of motor, η_t efficiency of pump, η_P ($\eta_t\eta_P$ = 0·25)
			110,000		27,500‡	industrial hydrogen peroxide
	solid fuel +motor and pump		200,000‖		50,000‡‖	combustion
	kerosene oil +motor and pump		4,500,000		1,125,000‡	combustion (weight of oxygen not included)
Potential	water power	H	~1,200			H = height of the fall, in m (La Bathie, Savoie)

* Values for the Air Equipment accumulator No. 30065 (accumulator for engines): Weight empty, 1·200 kg; 530 cm³ of air at 300 kg/cm², 0·200 kg; 270 cm³ of oil, 0·230 kg. When greater endurance is required, it is necessary to increase the coefficients from 6 to 12 for K_1 and from 4 to 8 for K_2.

† Air Equipment pneumatic motor No. 28,300 (power 3·5 kW, weight 1 kg).

‡ These figures take into account the weight of the motor and pump unit, which varies considerably according to type, size and endurance required. The following are a rough guide: hydraulic pumps, 0·18–0·40 kg/kW; pneumatic motors (air or gas) 0·25–0·60 kg/kW; electric motors 0·70–2·0 kg/kW.

§ These values correspond to a discharge time of about 1 h; if shorter, they must be divided by a factor of ca. 3 for 6 min, ca. 12 for 1 min. For a comparison between different types of electric accumulators, see e.g. Linden and Daniel[1].

‖ **Using the 'cold' solid fuels (1,400°K).**

where α is the ratio of the weight of the accumulator to the energy stored, which varies considerably with the type of accumulator, and β is the ratio of the weight of the supply unit to the power, which depends on the nature of the unit.

It is obviously impossible to give precise values of α and β, since in every case these coefficients depend on the technological quality of the different parts, the durability and degree of safety required, and the magnitude of the total energy to be stored, the maximum hydraulic pressure and the maximum rate of flow.

Table 2.1 gives a comparison between the different types of accumulators and sources of energy. In view of the numerical values quoted, it will be appreciated that: (*a*) accumulators with springs are rarely used, and then for an operation which need only be carried out once (e.g. releasing a safety device); (*b*) pneumatic accumulators which are not recharged while operating, are used continuously only in machines whose active life is limited to a time of about 10 sec; (*c*) electric accumulators which cannot be recharged are used only for a duration of a few minutes; for longer duration, an external source of energy must be employed, such as a mechanical pump to recharge a pneumatic accumulator, a generator to recharge the electric batteries, etc.

2.4. DISSIPATION OF ENERGY IN THE LIQUID ITSELF

In a hydraulic system, the energy in the liquid is either transmitted to the surroundings or dissipated as heat in the liquid itself.

The part transmitted outside is large in systems such as hydraulic transmissions (about 80 per cent) but small in control circuits. Often it is almost zero, namely when the external forces consist only of inertia and restoring forces (see Chapters 6 and 7).

The other part heats up the fluid and also the various parts of the circuit, but when there is no radiator it is dissipated mainly in the fluid itself.

It is advisable to estimate the increase in temperature of the fluid.

Heat produced from pressure—Let

W = weight of liquid
w = specific weight
c = its specific heat
J = the mechanical equivalent of heat.

The energy in the liquid is:

$$E_L = \Delta P\, V = \Delta P \frac{W}{w}$$

Energy required to raise its temperature by 1°C:

$$E_R = JWc$$

$\therefore$ Increase in temperature:

$$\Delta T = \frac{E_L}{E_R}$$

i.e.

$$\Delta T = \frac{\Delta P}{Jcw}$$

With

ΔP in kg/cm^2
W in g/cm^3
c in cal/g °C
ΔT in °C
J in kg/cal (0·426):

$$\Delta T = \frac{\Delta P}{100\ Jcw} = 0{\cdot}0235\ \frac{\Delta P}{cw}$$

For the usual hydraulic fluids, with $w \simeq 0{\cdot}86$ and $c \simeq 0{\cdot}5$

$$\Delta T = 0{\cdot}055\ \Delta P_{(\mathrm{kg/cm^2})}$$

Note—In practice, the energy dissipated in the pump also heats up the fluid. To find the total heat produced *per cycle*, the value obtained above must be divided by the efficiency of the pump, η_P. Thus, for example, in a circuit fitted with a 300 kg/cm^2 pump with an efficiency of 0·8, the maximum temperature rise per cycle is given by

$$\Delta T = 20{\cdot}6\,^\circ\mathrm{C}$$

Because of the danger of overheating*, some hydraulic systems are equipped with variable flow pumps.

REFERENCE

[1] Linden, D. and Daniel, A. F. *Electronics* **31** (1958) 59

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* Wherever possible, the operation of hydraulic circuits at high temperatures is avoided, because of the danger of fire; the increase in the rate of corrosive action and the decrease in the working life of the fluid and seals; and the detrimental effect on the performance, due to a reduction in the bulk modulus, B (see Section 5.3.2).

It is inadvisable to use the usual hydraulic fluids above 110–120°C in an open circuit or above 140–150°C in a closed circuit. When it is impossible to keep the operating temperature low (e.g. in a supersonic aircraft), special fluids such as Oronite FH 8 have to be used; while they can be employed up to 200°C, these fluids impose certain severe restrictions: storage away from air and water vapour; certain metals have to be avoided in the construction of the circuit; special seals, etc. Much research is being done on the development of fluids for use at high temperatures. Some workers have even suggested the use of molten metals!

3

HYDRAULIC CHARACTERISTICS

3.1. THE CONCEPT OF HYDRAULIC CHARACTERISTICS

It has been shown in Chapter 1 that for a simple hydraulic component or, more precisely, for an *undeformable non-active component with single inlet and outlet*, there will be a loss of head, ΔP, corresponding to a volume flow, Q, given by

$$\Delta P = \xi \frac{w}{2g} \frac{Q^2}{S_R^2}$$

This equation is the hydraulic characteristic of the simple component considered.

The operation of any hydraulic component can, in general, be expressed by an equation with more than two variables: α, β, γ, δ, etc. These may be hydraulic (e.g. rate of flow, pressures at different points), mechanical (such as angular speed or control setting) or electrical (control current), etc. Thus we can write the equation

$$f(\alpha, \beta, \gamma, \delta ...) = 0 \tag{1}$$

The hydraulic characteristic is the relationship between the two primary parameters α and β, of which at least one is hydraulic, when the other parameters in eqn. (1) are kept constant, i.e.

$$f(\alpha, \beta, \gamma_t, \delta_t ...) = 0 \tag{2}$$

and the characteristic curve is its representation on the graph of α against β.

The primary variables, α and β, are obviously chosen with regard to the operation considered. Thus, for a pump, the parameters normally considered are the volume flow and the outlet pressure $[Q = f(P_0)]$, keeping constant* the inlet pressure $[P_1 = P_1]$, the speed of rotation and the capacity. Sometimes the primary parameters are the outlet and inlet pressures, or the volume flow and the inlet pressure or even the internal leakage flow and the inlet pressure. There are many components, in particular multiple components, i.e. those with multiple inlets and outlets, for which there are different families of characteristics, depending on whether we consider the main flow, the bypass flow or, in some cases, the servo flow. It is important, therefore, to be aware of the *relative* nature of the hydraulic characteristic.

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* Note that the characteristics depend on the values of the parameters which determine the state of the fluid (density, viscosity, temperature, the proportion of air dissolved in the liquid or mixed with it, etc.). So, for a pump, we shall consider flow-outlet pressure characteristics for a given inlet pressure, rotational speed and capacity.

The operation of a system is governed by the physical laws, but often the equations expressing the operation either partially escape us or are too complex to be assimilated. The concept of hydraulic characteristics allows us to *isolate* the particular variation in which we are interested, from this complexity.

A hydraulic characteristic can be calculated from theory, determined by experiment or estimated by means of an analogy. In fact, *graphical representation* enables us to combine the information gained from calculation, from experiment and from experience of a similar system.

The hydraulic characteristic is therefore a most useful tool for the study of steady flow in hydraulic circuits and, in particular, for matching components. It permits 'graphical calculations' which, however qualitative they may be in the course of establishing a rough estimate, become more accurate as the work progresses and as the characteristics of each component are defined.

Although the hydraulic characteristic, in principle, only represents the steady-state condition and does not give information for transient states, it is most useful for dynamic analysis, since it indicates the important points, such as the failure point and points of maximum and minimum slope.

We shall now consider examples of hydraulic characteristics and their use (see also Chapter 10, Example 3).

3.2. HYDRAULIC CHARACTERISTICS OF NON-ACTIVE COMPONENTS WITH SINGLE INLET AND OUTLET

3.2.1. FLOW COMPONENTS HAVING FIXED SHAPE

This type of component was discussed in Chapter 1. The ΔP, Q characteristic is almost parabolic, since the coefficient ξ in the equation

$$\Delta P = \xi \frac{w}{2g} \frac{Q^2}{S_R^2} \tag{1}$$

is practically constant.

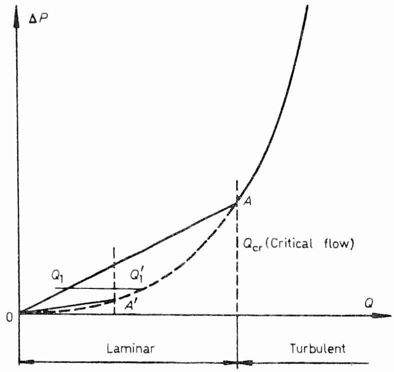


Figure 3.1

When the flow velocity, and hence the volume flow, decreases, the turbulent flow becomes laminar, and so the parabola is replaced by a straight line ($0A$) near the origin (since $\Delta P = KQ$, as shown in Section 1.2.4; see *Figure 3.1*).

Notes—(*a*) The laminar flow part of the characteristic is rarely used, except sometimes in studying the stability of hydraulic systems for which very small values of ΔP are considered. For those considerations, it is fortunate that $Q/\Delta P$ can be taken as constant, since this permits linearization of the equations. (In any case, if the parabola representing turbulent flow had been continued back to the origin, the ratio $Q/\Delta P$ would tend to infinity as ΔP tended to zero.)

(*b*) The characteristic curve shows the reduction in flow at low values of ΔP resulting from an increase in the viscosity of the fluid (falling from Q_1' to Q_1 when the viscosity changes from ν' to ν).

3.2.2. DEFORMABLE COMPONENTS

In deformable components such as valves, the cross-sectional area of flow at the restriction, S_R, is not constant and depends on certain hydraulic variables. We shall consider two examples.

3.2.2.1. *Relief valve* (*pressure-controlling valve*) (*Figure 3.2*)

In the very simple relief valve, SR is a function of ΔP. Eqn. (1), which should be written

$$\Delta P = \xi \frac{w}{2g} \frac{Q^2}{[S_R(\Delta P)]^2} \tag{1'}$$

is no longer useful, since it does not constitute an equation explicit in ΔP.

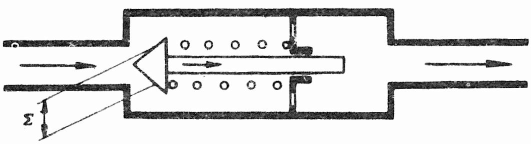


Figure 3.2

If we assume, as a first approximation, that the difference of pressure acts on an area Σ, against a spring of stiffness R, which is compressed with force F_0 when the valve is closed, and that the cross-sectional area, S_R, is related to the deflection x of the plunger by the equation $S_R = kx$, then the valve is closed for values of ΔP up to ΔP_0, where $\Delta P_0 = F_0/\Sigma$. For values of $\Delta P < \Delta P_0$, there is no flow, or only flow due to leakage.

For values of $\Delta P > \Delta P_0$, we have

$$\Delta P = \frac{F_0 + Rx}{\Sigma} \quad \text{or} \quad \Delta P = \Delta P_0 + \frac{Rx}{\Sigma}$$

Thus

$$S_R = \frac{k\Sigma}{R}(\Delta P - \Delta P_0)$$

and

$$Q = \sqrt{\frac{2g}{\xi w}}\, S_R \sqrt{\Delta P} = \sqrt{\frac{2g}{\xi w}}\, \frac{k\Sigma}{R}(\Delta P - \Delta P_0)\sqrt{\Delta P}$$

In general, for large values of x, $S_R \neq kx$ and, in particular, since the hydraulic force varies with the rate of flow, Σ is not constant*, so we have

$$Q = \sqrt{\frac{2g}{\xi w}\,\frac{k}{R}\,(\Sigma\,\Delta P - \Sigma_0\,\Delta P_0)}\;\sqrt{\Delta P}$$

It is sufficient to indicate here the general shape of the characteristics of valves for raising pressure and to point out that the useful part of the characteristic is, for most of the time, essentially a straight line (*Figure 3.3*).

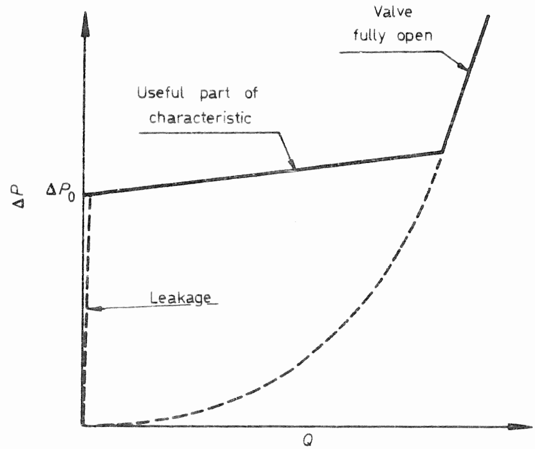


Figure 3.3

A *flat* characteristic is often required. Complete flatness is obviously impossible, since the spring would have to have zero stiffness. In fact, the limitation is not the difficulty in finding such a spring but the onset of instability due to the variation of the resultant force on the plunger (spring force + hydraulic force) which is a function of the position of the plunger and of ΔP. This problem will be dealt with below (Section 4.3.3).

Use of a relief valve—As a first example of the application of characteristic curves to a problem of matching (*Figure 3.4*), consider a hydraulic circuit in which the flow is subject to large variations. In this circuit, there is a component the correct operation of which requires that the liquid passing through it must at all times be above a certain minimum pressure, such as that necessary for the operation of internal controls.

If the characteristic of the downstream circuit does not allow the pressure to reach this minimum value for small rates of flow, then a pressure-raising valve must be placed at the outlet from the component. This is the case, for example, for fuel-regulating valves in jet engines where the fuel is injected at low pressure and vaporized before combustion, as against jet engines where the fuel is sprayed in under pressure.

* Σ is strictly only a coefficient having the dimensions of area and is defined as the actual total hydraulic force divided by ΔP.

The characteristic for the downstream part of the circuit including the relief valve (curve T) is found by adding the characteristic for the downstream part alone (curve A) to that of the valve alone (curve V). This enables us to

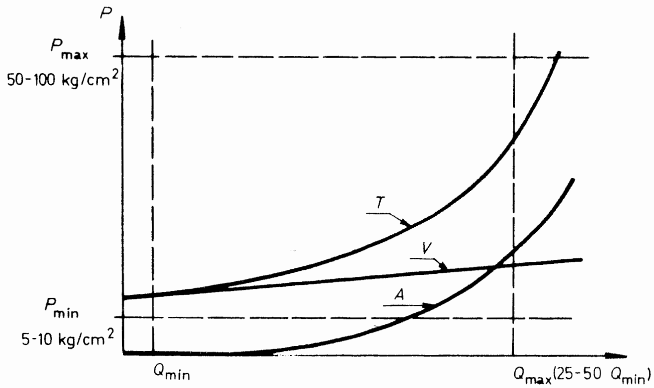


Figure 3.4

make a graphic analysis of the adaptation of the valve. The maximum and minimum values of P and the limiting values of Q can be estimated from the graph.

3.2.2.2. *Orifice with a linear characteristic*

As was shown, the volume flow in a relief valve is given by

$$Q = K\,(\Sigma\,\Delta P - \Sigma_0\,\Delta P_0)\,\sqrt{\Delta P}$$

It was also shown that Σ, the area on which the pressure difference acts, generally varies with the aperture. Assuming that the initial spring compression is zero ($\Delta P_0 = 0$) and that Σ varies as $1/\sqrt{\Delta P}$, the volume flow is given by

$$\boldsymbol{Q = K'\Delta P}$$

and the orifice has a linear characteristic.

In order to produce the hydraulic equivalent to an electrical resistance, two of these components must be combined so as to act for each direction of flow.

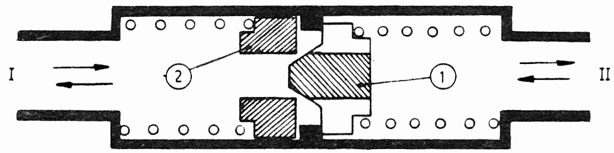


Figure 3.5

This combination is made in one unit (*Figure 3.5*). If the flow is from I to II, plunger (1) is depressed; if it is from II to I, plunger (2) is depressed.

3.3. HYDRAULIC GENERATORS CLASSIFIED ACCORDING TO TYPE OF CHARACTERISTIC

3.3.1. THE TWO MAIN TYPES OF HYDRAULIC GENERATORS

The hydraulic generators in general use are pumps and reservoirs under pressure. Reservoirs are really only accumulators, but they can be regarded as generators if the method of filling is not considered. From the point of view of their P–Q characteristics, however, i.e. considering their method of use (see Section 3.5.2), it is advisable to adopt a completely different classification, giving curves which cut across those of the pump group.

We can differentiate between pressure generators and flow generators.

Pressure generators are those capable of providing any flow under constant pressure; they have horizontal characteristics: P = constant.

Flow generators are those capable of providing constant flow under any pressure; they have vertical characteristics: Q = constant.

No generator, either of pressure or flow, is perfect, and the imperfections of each type will be specified below.

3.3.2. PRESSURE GENERATORS

3.3.2.1. *Reservoirs under pressure*

A reservoir under pressure constitutes the most perfect source of pressure. However, infinite flows are not possible and the horizontal characteristic must curve downwards towards the Q axis for large values of Q (see *Figure 3.6*). This is due to the loss of head at the outlet from the reservoir.

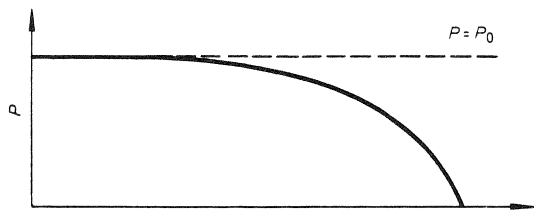


Figure 3.6

The problem presented by accumulators under pressure is that of maintaining the pressure applied to the liquid despite the variations in volume caused by the fluid discharged. The simplest solution is to fill a portion, λV_t, of the total volume, V_t, of the accumulator with a gas at pressure P_1, the volume of the stored liquid not exceeding $(1-\lambda)V_t$ and its pressure decreasing from P_1 to λP_1. In practice, the two fluids are separated either by a piston, known as a free piston—which means that the accumulator must be of cylindrical shape—or by a rubber diaphragm, which is more suitable for a spherical shape; this, in turn, being convenient for reasons of strength. For economy of space, we can either decrease λ by supplying the volume occupied by the gas from a reservoir at a very high pressure and expanding it to the pressure required or even by using the products of combustion from a cartridge; or else, we can reduce the total volume by supplying the liquid from a pump which is started automatically when the pressure is lowered to a certain value or when the free piston reaches a certain position. If the flow from the reservoir is variable, the pump can operate permanently. This flow, then, is controlled either by the pressure of the gas or the position of the free piston.

Such measures are necessary to preserve the principal quality of the accumulator, which is its insensitivity to *abrupt variations of flow*. This quality alone often justifies its presence as a buffer between the pump and the rest of the circuit. For example, the stopping time for the pump supplying the accumulator will determine the minimum volume of gas necessary for the gas pressure not to exceed its upper limit when there is an abrupt fall in the demand from the accumulator, and vice versa.

An accumulator with a diaphragm reponds better to abrupt variations in the flow than a piston accumulator because of the inertia of the piston itself.

3.3.2.2. *Hydrodynamic Pumps*

Hydrodynamic pumps (centrifugal, axial and helical) are also sources of pressure. Their discharge pressure is approximately a function of their rotational speed, N, only (proportional to N^2), and we shall assume this to be constant.

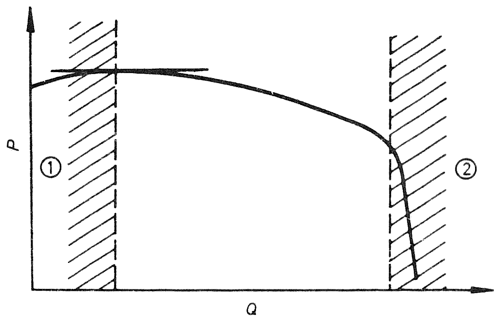


Figure 3.7

In fact, however, it is a little more complex* (cf. Chapter 10, Example 3), since

(*a*) according to Euler's equation, the energy transmitted to the liquid is proportional to its tangential velocity, V_t, at the exit from the rotor—but V_t is always a decreasing function of the flow;

(*b*) the efficiency is a function of the relative angle between the trailing edges of the blades on the rotor and the fluid leaving the rotor, and also on the shape of the diffuser, being a maximum for a certain value of Q/N;

(*c*) at the suction side of the pump, any increase in speed will cause a reduction in pressure (proportional to Q^2) which, if excessive, may be lowered to the vapour pressure of the liquid, resulting in cavitation—obviously the inlet pressure cannot be lowered further;

* See, e.g., A. de Kovats, *Pompes, ventilateurs, compresseurs centrifuges et axiaux*, Dunod, Paris; cf. Chapter 10, Example 3.

(*d*) the various component parts of the pump will each cause their own individual loss of head.

Because of these effects, the characteristic curve takes the form shown in *Figure 3.7*.

The descending portion of the curve may be affected by a variation of the inlet pressure. (In modern centrifugal pumps for after-burning or ram jets, P can reach 60–80 kg/cm^2 and Q 20–30 l./sec). Operation in regions 1 and 2 should be avoided because, in region 2, there is an abrupt fall in pressure and a possibility of abrupt variations in torque and, in particular, because of the danger of cavitation in the pump; in region 1, the positive slope of the curve indicates considerable risk of instability. We must also avoid overheating, since the flow in this zone is small and may be insufficient to cool the pump.

Protection of hydrodynamic pumps—To ensure that the pump is not allowed to operate in region 2, it must simply be made sufficiently large. To ensure, however, that it does not operate in region 1 if the flow varies from 0 to Q_{max}, it is necessary to protect the pump.

The simplest form of protection consists in placing an orifice in parallel with the pump to permit sufficient flow at all times. The characteristic T of the pump equipped with bypass orifice is easily deduced from that of the pump alone, P, and the orifice alone, O (for each value P_i of P, $AC = AD - AB$; see *Figure 3.8*).

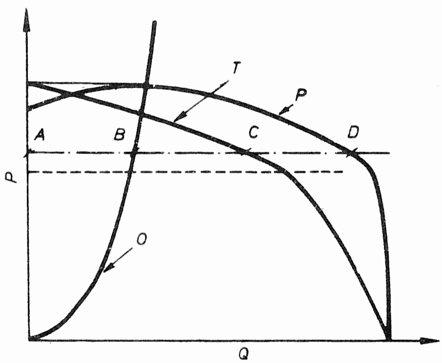


Figure 3.8

Note that the pressure relief valves used to protect volumetric pumps are useless for hydrodynamic pumps.

3.3.3. FLOW GENERATORS

This category consists of *positive displacement pumps* (gear pumps, piston pumps, vane pumps) whose characteristics have the form $Q = KN$ (provided that they do not include a mechanical device for changing the displacement volume).

In practice, due to turbulent and laminar internal leakage and to changes of shape and mechanical deformations at high pressure, the characteristic curve takes the form $Q = KN - f(P)$, as shown in *Figure 3.9*.

When the orifice at the outlet of a good positive displacement pump is gradually closed, the delivery pressure builds up so as to eventually cause mechanical failure of the pump itself. *It is therefore necessary to protect a positive displacement pump from high pressures, just as it is necessary to protect a hydrodynamic pump from low pressures.*

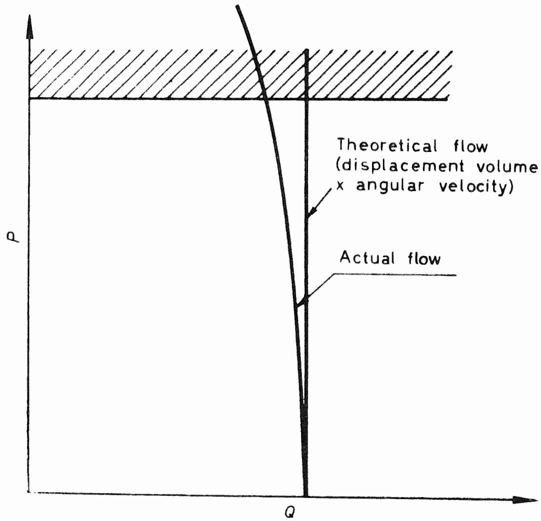


Figure 3.9

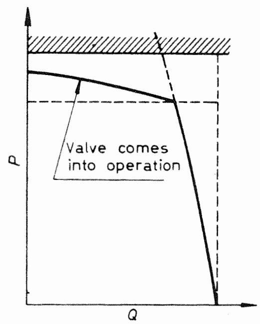


Figure 3.10

The pressure limits necessary to ensure a good length of life are:

Type of pump	*Pressure limit*	
	kg/cm^2	lb./in.^{2}
Rotary spur gear	75–125	1,000–2,000
Internal gear (lateral forces balanced, or special bearings	125–225	2,000–3,000
Piston, for fuel (petrol, JP4, kerosene)	150–200	2,000–3,000
Piston, for oil and hydraulic fluids	200–300	3,000–4,000
Piston (under development)	450–500	6,000–7,000

Protection of positive displacement pumps—The simplest device to avoid the build-up of pressure referred to is a pressure-relief valve mounted in parallel across it (see Section 3.5.3). The characteristic of a pump fitted with such a valve takes the form shown in *Figure 3.10*.

3.3.4. COMBINATIONS OF PUMPS

When it is impossible to purchase a pump which will supply either the volume flow or the pressure required, pumps can be combined, either in parallel (adding the flows) or in series (adding the pressures).

A consideration of the characteristics shows (and practical experience confirms) that, in principle, *connections in parallel should be confined to positive displacement pumps* (*Figure 3.11*), *connections in series to hydrodynamic pumps* (*Figure 3.12*).

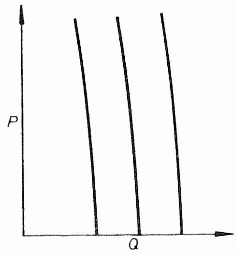


Figure 3.11

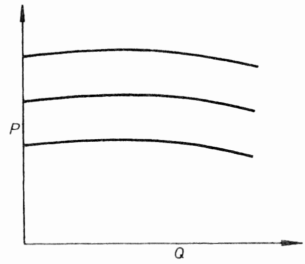


Figure 3.12

Notes—(*a*) A positive displacement pump equipped with a pressure-relief valve can be considered as a source of pressure, as has been shown. It can thus be connected in series with another positive displacement pump, provided that it is placed upstream and its nominal volume flow is greater than that of the other pump. This arrangement is often used to improve the supply to a high-pressure pump, a booster pump being placed in the supply pipe (*Figure 3.13*).

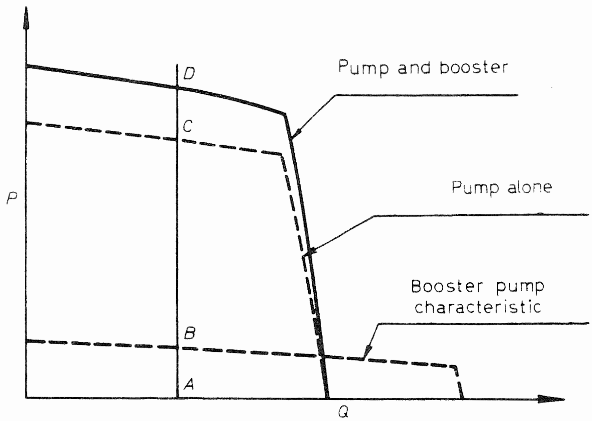


Figure 3.13

(*b*) The practical impossibility of connecting positive displacement pumps in series and hydrodynamic pumps in parallel does not often lead to difficulties. Positive displacement pumps are suitable for giving small flows at high pressures —the increase in flow increases the size of the pump, and the efficiency decreases relatively little with pressure—and hydrodynamic pumps are suitable for giving large flows at moderate pressure—the increase in flow has no great effect on the overall dimensions and usually increases the efficiency, but the pressure rise is limited by mechanical (angular velocity) or hydraulic considerations (cavitation).

3.4. CHARACTERISTICS OF HYDRAULIC CONTROL COMPONENTS

3.4.1. HYDRAULIC CONTROL COMPONENTS

All hydraulic *receivers* consist of one or more chambers in which the liquid does a certain amount of work against external forces. For this reason they are called work chambers or work volumes.

The function of a hydraulic control component is to regulate the admission of the liquid into these chambers.

In certain cases, for example in an open chamber such as in turbines, there is only the problem of regulating the input of liquid, and a tap or cock is sufficient. In other cases, where the chamber is closed, such as in a ram or actuator, both the input and output of the liquid must be regulated. There will be two pressure sources, 1 and 2, at pressures P_1 and P_2 which will be higher than the maximum pressure in the work chamber and lower than the minimum pressure, respectively.

The second type of control component (*Figure 3.14*) will be considered here.

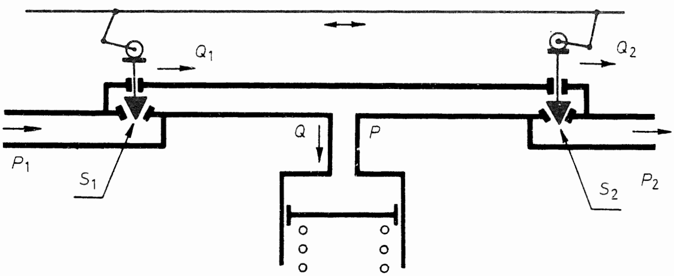


Figure 3.14

Two valves of cross-sectional area of flow, S_1 and S_2, control the entry and exit of the liquid, respectively. If

ξ_1, ξ_2 = the coefficients of loss of head of the valves
Q_1, Q_2 = the flow through them
Q = resultant flow into the work chamber
P = pressure in the work chamber

we can form three basic equations from the 10 parameters $\xi_1, \xi_2, S_1, S_2, Q_1, Q_2, P_1, P_2, P$ and Q:

$$P_1 - P = \xi_1 \frac{w}{2g} \frac{Q_1^2}{S_1^2} \tag{1}$$

$$P - P_2 = \xi_2 \frac{w}{2g} \frac{Q_2^2}{S_2^2} \tag{2}$$

$$Q_1 = Q_2 + Q \tag{3}$$

But ξ_1 and ξ_2 are usually constant (if not, they depend on S_1 and S_2, or rather on Q_1 and Q_2). P_1 and P_2 are constant and depend on Q_1 and Q_2 and on the upstream and downstream P–Q characteristics. S_1 and S_2 depend on the position of the control, e (although one of the areas S_1 and S_2 may be constant): sometimes they depend on e and on P.

This leaves five variables: e, Q_1, Q_2, P and Q. The elimination of Q_1 and Q_2 from the three equations gives a relationship between e, P and Q. In order to define the state of the system, one more equation is required. This is provided by the relationship between P and Q, which can be found knowing the load applied to the work chamber.

The operation of the control therefore depends on the impedance of this load, hydraulic impedance being defined as $\Delta P/\Delta Q$, i.e. the slope of the characteristic.

Two extreme cases in particular are interesting because of the easy analysis and because they constitute a very good approximation to many real components:

(1) where the impedance is very large, Q is in fact reduced to the compressibility flow and is negligible compared with Q_1 and Q_2. Control components which operate against high impedance, for which we can assume $Q = 0$, are called *hydraulic potentiometers*.

(2) where the impedance is very small, variations of ΔP and P are small compared with $P_1 - P_2$, and we can assume that $P = P_0$ is constant. Control components which operate against low impedance, for which we can assume $P = P_0$, are called *hydraulic control valves*.

Note that, in the course of the analysis, some apparatus will be classified first in one category and then in the other. For example, the valve of an actuator controlling an aerodynamic control surface, such as a rudder or aileron, must be considered as a control valve for the analysis of dynamic performance at high frequency and small amplitude, and for stability and no-load analyses, but as a potentiometer in the analysis of static performance.

3.4.2. THE HYDRAULIC POTENTIOMETER*

The hydraulic potentiometer has been defined as a control component with a high impedance, $Q \simeq 0$. Hence eqn. (3) becomes $Q_1 = Q_2$. Eliminating Q_1 and Q_2 from eqn. (1) and (2) gives the characteristic equation of the hydraulic potentiometer:

$$\frac{P - P_2}{P_1 - P_2} = \frac{\xi_2/S_2^2}{\xi_1/S_1^2 + \xi_2/S_2^2}$$

i.e.

$$\boxed{\frac{P - P_2}{P_1 - P_2} = \frac{1}{1 + \xi_1 S_2^2/\xi_2 S_1^2}} \tag{4}$$

and, if we can assume $\xi_1 = \xi_2$

* See also Sections 5.2.3, 5.4.2.3 and 6.4.5, Example 1.

$$\boxed{\frac{P - P_2}{P_1 - P_2} = \frac{1}{1 + (S_2/S_1)^2}} \tag{4'}$$

This relationship is similar to the characteristic equation of the electrical potentiometer, and the diagrams are also similar (*Figure 3.15*). Note, however, that instead of the ratio of the electrical admittances, there is the *ratio of the squares of the areas of the valve openings*. This fact, which is due to the non-linearity of the relationship between loss of head and flow, is the root of the differences in behaviour between hydraulic and electrical circuits. It results in the high sensitivity of the hydraulic potentiometer in the region $S_1/S_2 = 1$ and the low sensitivity when S_1/S_2 is either small or large (see *Figure 3.16*). In a more general way it explains how hydraulic circuits may be easily choked when some areas of the flow passage are slightly undersized, and also the difficulty in making a hydraulic circuit operable over a wide range of flow.

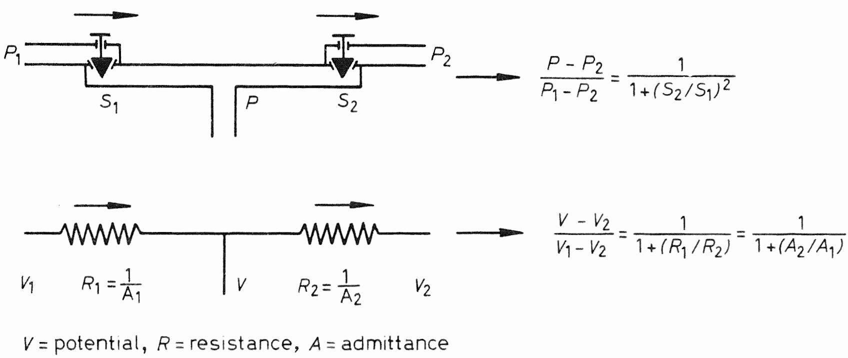


Figure 3.15

Examples of hydraulic potentiometers—The hydraulic potentiometer is widely used in hydraulic systems. Despite considerable differences in design and operation, all are governed by eqn. (4). We shall describe two types which deviate slightly from the classical models.

1. *Flapper-and-nozzle valve*

Consider a hydraulic potentiometer with a fixed orifice, S_1, and a variable orifice, S_2, consisting of a nozzle partially obstructed by a flapper plate whose plane is parallel to the plane of the orifice (*Figure 3.17*). The regulation parameter is e, the distance of the plate from the orifice, S_2.

Numerical application—Suppose that the orifices, S_1 and S_2, have very small diameters, say 0·5 and 1·0 mm, respectively, in order to keep the volume flow small. We have to calculate the variation Δe of e necessary to change P from 50 to 55 kg/cm^2, for $P_1 = 100$ kg/cm^2 and $P_2 = 0$.

For $(P-P_2)/(P_1-P_2) = 0{\cdot}50$ and $0{\cdot}55$, we have $S_2/S_1 = 1$ and $0{\cdot}9$ from *Figure 3.16*. In this case ($S_1 = 0{\cdot}2$ mm^2), $S_2 = 0{\cdot}20$ and $0{\cdot}18$ mm^2 and, finally, $e = S_2/\pi D_2 = 0{\cdot}064$ and $0{\cdot}0575$ mm, giving $\Delta e = 0{\cdot}0065$ mm.

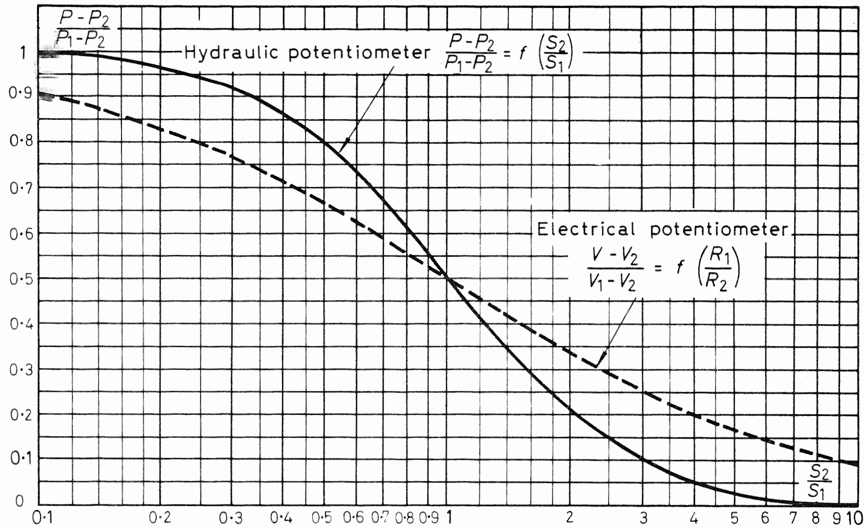


Figure 3.16

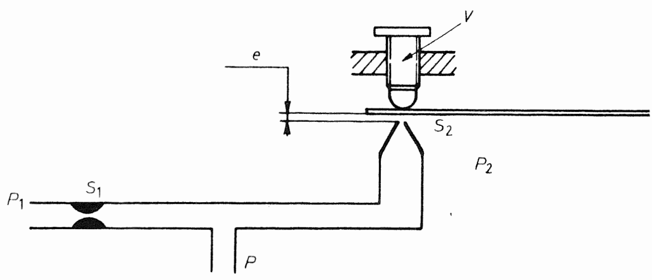


Figure 3.17

Precise regulation is extremely difficult. (If the pitch of the thread on the adjusting screw is $0{\cdot}5$ mm, Δe represents a rotation of less than 5°.) For this reason, a spring is often placed between the screw and the plate (*Figure 3.18*). The position taken by the plate is the result of the equilibrium between the force exerted by the spring, $K(l-l_0)$, and the force from the liquid, $P\sigma_2$($\sigma_2 = \pi D_2^2/4$ being the area of the orifice O_2). For $P = 50$ and 55 kg/cm^2, $P\sigma_2$ equals 400 and 440 g.

By selecting a spring of stiffness $0{\cdot}1$ kg/mm (10 mm for 1 kg), an additional force of 40 g is equivalent to a compression of $0{\cdot}4$ mm. Since the plate must be displaced by 0·0065 mm, as we have seen, the new displacement necessary is 0·406 mm, which corresponds to almost a complete turn of a screw thread of pitch 0·5 mm.

We shall see in the course of the dynamic analysis that this replacement of a position input by a force input modifies not only the gain but also the nature of the transfer function.

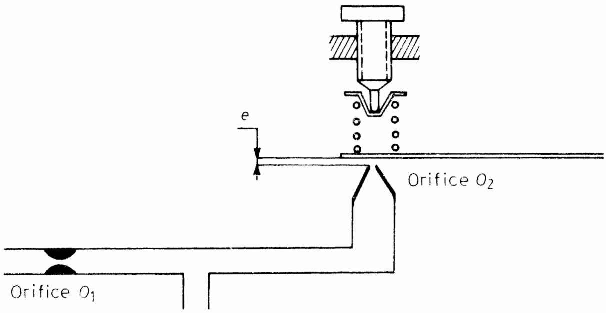


Figure 3.18

2. *Flow regulation with barometric pressure correction potentiometer*

This potentiometer is attached to the flow regulator shown in *Figure 3.38* (p. 79), in order to vary the flow, Q, as a function of some external parameter. For example, it can regulate according to the equation $Q = Kp$, where p is the atmospheric pressure. The restricting orifice S_1 (fixed) is mounted between the two chambers of the by-pass valve, and the orifice S_2 (which varies with p) is mounted at the outlet from the second chamber (*Figure 3.19*).

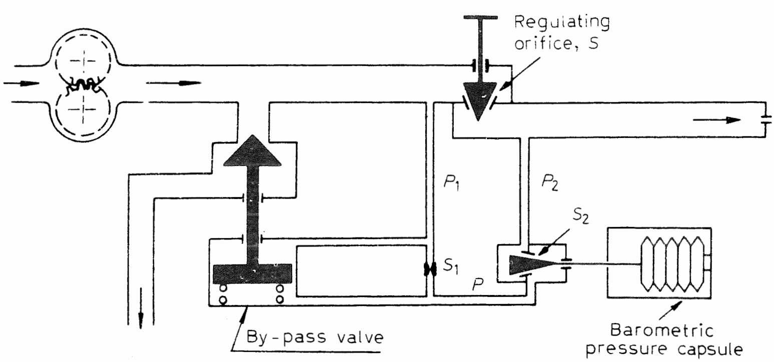


Figure 3.19

The by-pass valve maintains a constant pressure difference, so that we have $P_1 - P = K_1$ and, since the flow provided is proportional to the square root of the loss of head through the orifice S, we have:

$$Q = K_2\sqrt{P_1 - P_2}$$

Eqn. (4′)

$$\frac{P - P_2}{P_1 - P_2} = \frac{1}{1 + (S_2/S_1)^2}$$

can be rewritten

$$P_1 - P_2 = (P_1 - P)\,[1 + (S_1/S_2)^2]$$

from which

$$Q = Kp = K_3\sqrt{1 + (S_1/S_2)^2} \tag{5}$$

The barometric pressure capsule has a given pressure-displacement characteristic, in general $x = K_4 p$. It is straightforward, then, to combine this with eqn. (5) to give the relationship between the position of the pressure capsule and the flow area of the orifice S_2.

The possibilities of the equipment can be seen by simply plotting the curve of $X = \sqrt{1 + (1/r^2)}$ where $r = S_2/S_1$. In particular, in order to have a wide range of flow for a reasonably small range of r, it is advisable to use values of $r < \frac{1}{2}$—a region in which X is approximately equal to $1/r$ (*Figure 3.20*).

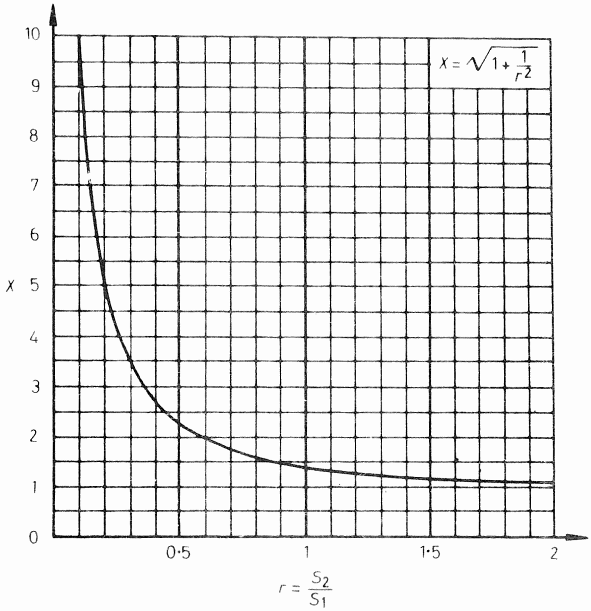


Figure 3.20

Pneumatic potentiometer—Pneumatic potentiometers are often used in hydraulic equipment, particularly in fuel regulators. Qualitatively, their operation is the same as that of the hydraulic potentiometer. Quantitatively, as the pressure ratio increases from unity, eqn. (4) becomes invalid owing to the compressibility of the air.

The theory for pneumatic potentiometers must necessarily include thermodynamic considerations, i.e. we must take into account not only the parameter P_1/P_2 but any exchange of heat with the surroundings.

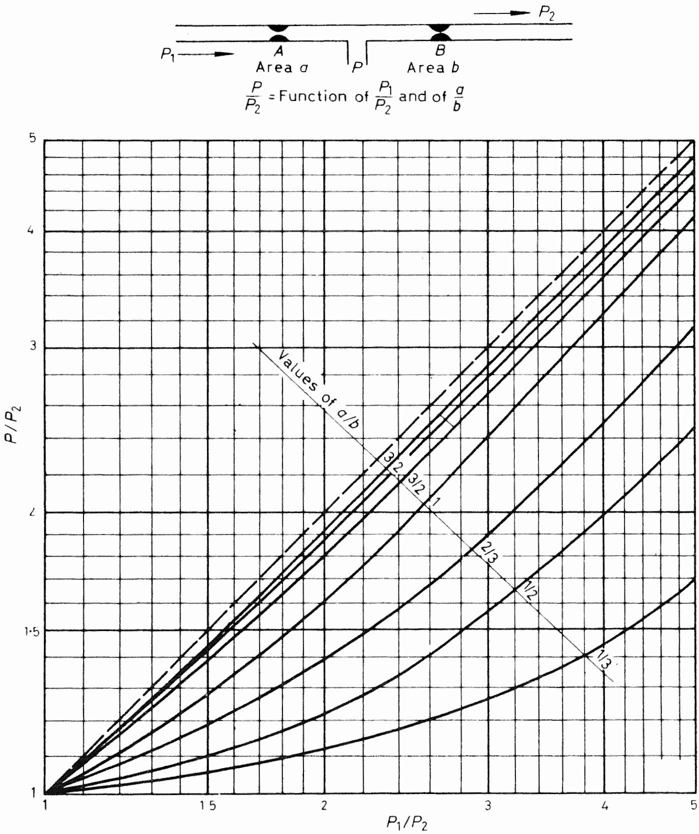


Figure 3.21. Pneumatic potentiometer

As an example, consider the experimental curves from an existing apparatus (a potentiometer for the control of a pneumatic Machmeter) shown in *Figure 3.21*. The accuracy of these curves is adequate for initial design estimates.

3.4.3. HYDRAULIC CONTROL VALVE

The hydraulic control valve has been defined as a control component with a low load impedance

$$\Delta P = 0, \qquad \text{i.e. } P = \text{constant}$$

Substituting this in the basic equations (1)–(3), we have

$$Q_1 = K_1 S_1 \tag{6}$$

$$Q_2 = K_2 S_2 \tag{7}$$

$$Q = Q_1 - Q_2 = K_1 S_1 - K_2 S_2 \tag{8}$$

In order to simplify the analysis, and without detracting in any way from the validity of the results, we shall examine the symmetrical valve shown* in *Figure 3.22* with $P = (P_1+P_2)/2$.

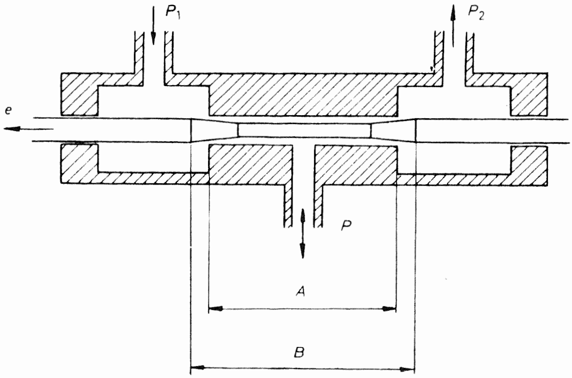


Figure 3.22

The variations of flow, Q_1 and Q_2, with the translational movement e are shown in *Figure 3.23*.

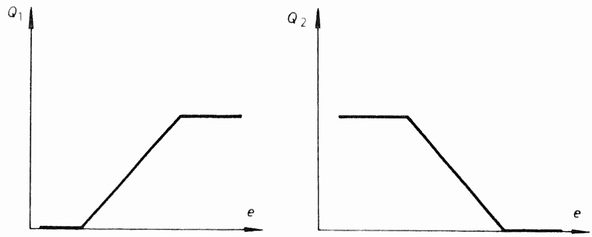


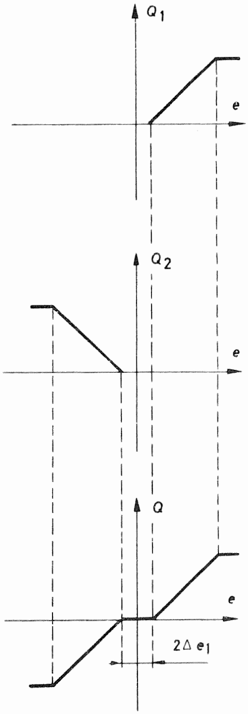
Figure 3.23

The variation of Q with e depends upon the relative setting of the profiles of the valve control spool. If, for a certain range of $e = 2\Delta e_1$, the cross-sectional areas, S_1 and S_2, are both zero, Q is as given in *Figure 3.24*. The valve is said to be overlapped, and there will be a dead zone.

If, for a certain range of $e = 2\Delta e_2$, the valves S_1 and S_2 are both slightly open, Q is given by *Figure 3.25*. The valve is said to be underlapped, and there will be internal leakage.

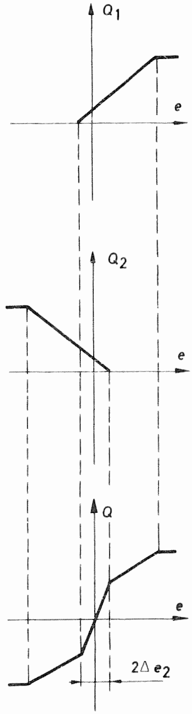
If S_1 begins to open exactly when S_2 closes, Q is given by *Figure 3.26*. This last type, where there is neither over- nor underlap, is often required but is difficult to achieve in practice, since it requires exact equality of the lengths A and B (see *Figure 3.22*). In the manufacture of electrohydraulic servo valves,

* The diagrams shown represent valves controlled by translational movement. There are also valves controlled by rotational movement[1].



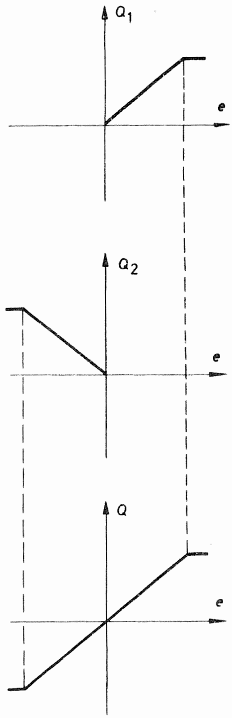
Overlap

Figure 3.24



Underlap

Figure 3.25



Zero lap

Figure 3.26

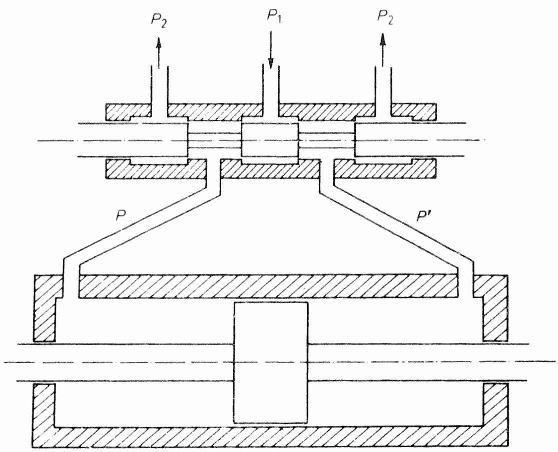


Figure 3.27

for instance, it is done by alternately machining down the part and testing the flow through the component.

Double valves are often used in order to supply the two chambers of a ram (*Figure 3.27*). In this case, the problem is to match three dimensions instead of one.

The overlapped valve is often used to improve the stability of the equipment connected to it. This improvement is made at the cost of accuracy. The angles of the curve of Q against $f(e)$ are rounded off due to leakage, so that Q is not zero for the whole interval $2\Delta e_1$ (*Figure 3.28*). This will be dealt with in Section 7.6.2.2.

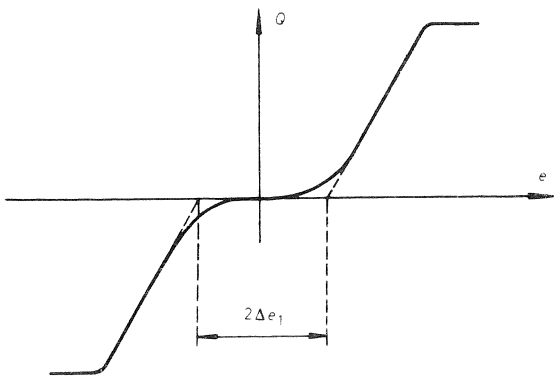


Figure 3.28

The underlapped valve is also used to improve stability. This may seem paradoxical. In fact, the improvement in stability is achieved despite the increase in gain (which is doubled near the origin), owing to the permanent throttling of the liquid which reduces the total energy by frictional losses (see Section 7.7.1).

The local increase in gain can be cancelled by modification of the control spool profile. This modified spool is called a double-sloped spool; the effect of the two slopes is shown in *Figure 3.29*.

Unfortunately, the underlapped valve allows a slight flow of liquid even when $e = 0$, and this fact often renders it unsuitable for aeronautical applications.

The gain of a control valve—The gain of a control valve is the ratio $\Delta Q/\Delta e$, i.e. the slope of the characteristic $Q = f(e)$. As we shall see later, it is a fundamental parameter in the analysis of hydraulic circuits which include these valves. It can be changed by altering the profile of the valve spool.

The simple classification of hydraulic control components into potentiometers and control valves is not strictly applicable for the dynamic theory, as will be seen later. Nevertheless, it represents a good method of approach to problems of flow control.

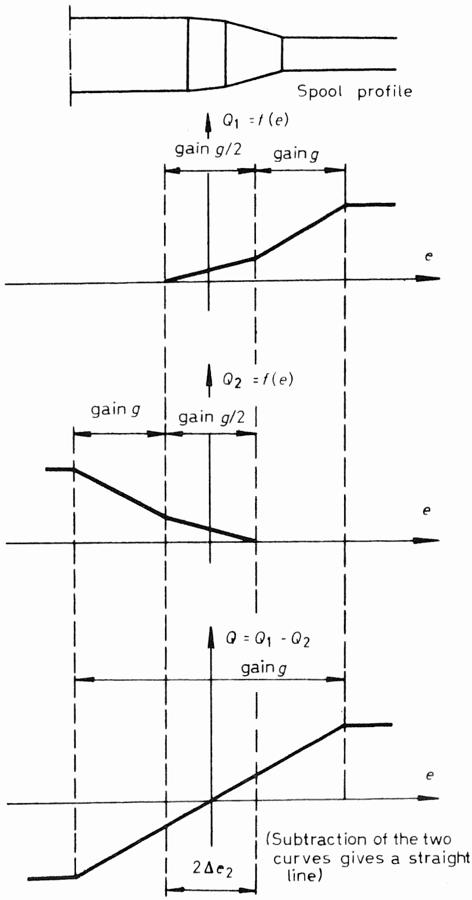


Figure 3.29

3.5. REGULATION OF PRESSURE OR FLOW

3.5.1. ADJUSTMENT AND REGULATION

It is important to differentiate between the meanings of these words. We will define: *adjusting components* (of pressure or flow) as all components controlling pressure or flow by the action of one or more intermediate parameters (cross-sectional area of a restricting orifice, angular velocity of a pump, displacement volume etc.) which are independent of the value effectively achieved.

We will define *regulation components* or *regulators* as such components where the *value* of the pressure or flow *effectively obtained* is measured and used to *control* the operation, in such a way as to nullify or at least reduce the effect of the perturbations of the circuit which affect the relationship between the intermediate regulation parameter and the pressure or flow.

The preceding Section was concerned with certain adjusting components. This one will deal with some regulators, i.e. servo controls. For reasons of clarity, these regulators will sometimes be represented by their block diagrams; these can all be reduced to the basic block diagram (*Figure 3.30*) on which appear the three basic components. These are:

(*a*) the adder which establishes the error, i.e. the difference between the input and the output;
(*b*) the working component which provides the output from the error, and
(*c*) the detector which measures the output and feeds this measurement back to the adder.

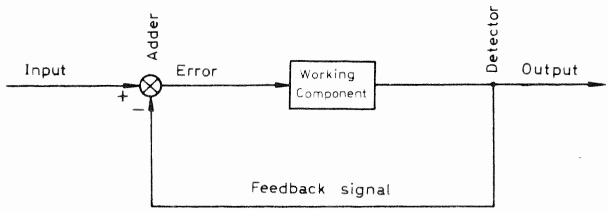


Figure 3.30

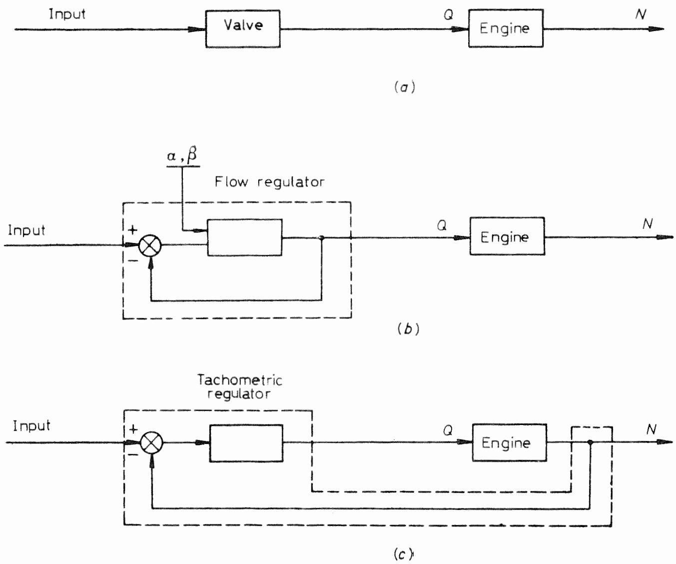


Figure 3.31

Notes—(1) Some adjusting components, especially when they are controlled by external parameters, such as pressures or temperatures, are often called regulators. In addition, we sometimes specify regulators as '*open-chain*' or '*open-loop*' to distinguish them from '*closed-chain*' or '*closed-loop*' regulators, referring to the form of the servo system as shown in the block diagram.

(2) A true closed-loop regulator can constitute a simple component in an adjusting chain without causing the chain to be a regulator. For example, to

control the rotational velocity, N, of an internal combustion engine, we can
(*a*) put a valve in the fuel supply line to the combustion chamber. The valve is an adjusting component for the flow, Q, of fuel and therefore it is also an adjusting component for the speed of rotation, N (*Figure 3.31a*);
(*b*) replace the valve by a *flow regulator*, controlled either manually or by an external parameter (ambient pressure, flight velocity, etc.). We will have freed the motor from variations of pressure in the fuel supply, but the complete loop is still a loop adjusting the speed of rotation, N, whatever the number of external parameters introduced into the flow regulator through secondary inputs (*Figure 3.31b*);
(*c*) install a tachometric regulator controlling the flow, Q, of fuel by the effective speed of rotation, N, of the engine: this illustrates true regulation of N (*Figure 3.31c*).

3.5.2. THE COMPONENTS OF A PRESSURE OR FLOW REGULATOR

The detector—The detection of pressure is never a problem. The detection of flow is usually made through the intermediary of pressure difference (or of a single pressure*). Thus, in modern regulators for controlling fuel supply, we detect the difference in pressure between the upstream and downstream sides of a calibrated orifice (the cross-sectional area of which also provides an excellent regulating parameter).

The adder—It is easy to apply the pressure (or pressure difference) to a membrane, capsule or piston in order to exert a *force* to act against another force provided by the control (the tension in a spring, for example).

The working component—The working component transforms the order sent to it by the adder into an *action upon the flow*, which can be
(1) a throttling of the main duct
(2) the opening of a diversion or by-pass
(3) a change in the speed of rotation or displacement volume of the generator, if it is a pump
(4) a combination of these operations.

The choice of mode of action of the working component is not arbitrary, however; it depends on the type of characteristic of the generator (see Section 3.3).

(1) The *throttling action* means that the flow used is equal in magnitude to the flow generated. It is not used with flow generators (positive displacement pumps) but is *used with pressure generators* (*accumulators and hydrodynamic pumps*).

(2) *Diversion of flow* means that the pressure at the entry of the downstream circuit is the same as that produced by the generator. It is not used with pressure generators but is *used with flow generators* (*positive displacement pumps*).

(3) *The change of speed of rotation* allows regulation of the *pressure* of a *hydro-*

* For a long time, in fuel regulators for jet engines only the upstream pressure in the injectors was measured. The desired flow was only obtained if the pressure-flow characteristic of the injectors was identical with the characteristic used in the design of the regulator. These regulators were affected by slight errors in manufacture, by clogging, by back pressure, etc. and should have been called pressure regulators rather than flow regulators.

dynamic pump or the *flow* of a *positive displacement* pump, but apart from its technical complexity it has the disadvantage of introducing a fairly high time lag due to the inertia of the pump rotor*.

The change of displacement volume allows regulation of the *flow* of a *positive displacement* pump with less complexity and a much lower time lag (which is almost negligible, 0·05 or even 0·02 sec). Unfortunately, this solution is only applicable to some positive displacement pumps.

(4) Several combinations of these methods of action could be envisaged. One example would be the flow regulation of a positive displacement pump by fitting it with a pressure relief valve and throttling the duct from the valve (the pump with the valve can then be regarded as a pressure generator). This solution (diversion + throttling) has the disadvantage of quickly wearing out the pump, since it operates constantly at its maximum speed and consumes and dissipates a constant rate of energy into the fluid, independent of the flow effectively used. However, it has been used for some time, and even now it is still employed occasionally. Another example would be: when the flow from a hydrodynamic pump has to be regulated over a large range, while avoiding operation at low rates of flow where there is often instability. This can be done by varying the speed of rotation, provided, of course, that the restrictive component is not omitted from the circuit. At the same time, the variation in speed or displacement volume can be used to provide the regulation of pressure, especially if turbulence in a large quantity has to be avoided, so as to reduce the energy dissipated, as in the case of a fuel pump for a jet engine at high altitude.

3.5.3. PRESSURE RELIEF VALVE, PRESSURE REDUCING VALVE, UNLOADING VALVE AND THROTTLING VALVE

The functions of the adder and the working component are frequently carried out by a single component. According to the nature of the control (pressure or pressure difference) and to the method used (by-pass or throttling), this component will have the name given in the following Table†.

Pressure-relief valve—The pressure-relief valve, whose function is to regulate the *pressure* of a *flow generator*, is a valve which raises the pressure (see Section 3.2.2) placed in a by-pass branch instead of in series with the generator (*Figure 3.32*).

There are two hydraulic parameters which affect it: $(P_p - P_b)$ and Q_b. Its characteristic is therefore the same as that of the pressure-raising valve if Q is replaced by Q_b (*Figure 3.33*). The variation of pressure with flow is not the theoretical horizontal straight line, because (*a*) the length of the spring decreases as the valve opens; (*b*) there are forces applied to the liquid by the valve (these

* A significant time lag in the flow regulator of a positive displacement pump may cause dangerous pressure rises during abrupt variations in demand. These pressure peaks are easily suppressed by the addition of an accumulator buffer.

† There is some controversy about this nomenclature. For example, the pressure-relief valve can be called an adjustment valve, a pressure valve, etc., the name pressure-relief valve being reserved for a *safety* valve which opens when the pressure rises to a certain value above normal working pressure, thus operating only under very exceptional circumstances. Likewise, the pressure-reducing valve is often called a pressure regulator, etc.

Method	Nature of control	
	Pressure regulation	*Flow regulation*
By-pass (generator: positive displacement pump)	pressure-relief valve	unloading (or by-pass) valve
Throttling (generator: accumulator or hydrodynamic pump)	pressure-reducing valve	throttle valve

hydraulic forces will be considered in Chapter 4). Knowing the characteristic of the valve, the by-pass pressure and the characteristic of the pump on its own, we can plot the characteristic of the combination of the pump and valve together (see also Chapter 10, Example 3.2). For each value of P_p deduct the by-pass flow, Q_b, from the pump flow, Q_p, to obtain the flow used, Q_u (*Figure 3.33*).

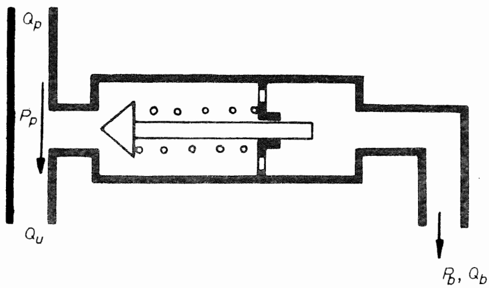


Figure 3.32

Pressure-reducing valve—The function of a pressure-reducing valve is to regulate the *pressure* of a *pressure generator*. It causes a loss of head in the circuit, such that the downstream pressure remains constant. All pressure-reducing valves can be represented basically by the diagram shown (*Figure 3.34*). The pressure, P_u, exerts a force, SP_u, on the piston. If P_u is increased, this force acts against the spring R and reduces the orifice at the valve inlet.

There are three hydraulic parameters which affect the valve: P_u, P_A and Q. For a perfect pressure-reducing valve, P_u would be constant. In the actual valve, P_u is a function of both P_A and Q, due to the effects of (a) the opening (the change in the spring force) and (b) hydraulic forces.

The displacement of the piston, x, is related to the area of the opening, σ, and therefore to the quantity $Q/\sqrt{P_A - P_u}$, by Bernoulli's equation.

In Chapter 4 it will be shown that, *for a given position of the piston unit*, the hydraulic force is proportional to $P_A - P_u$. *The characteristic of a pressure-reducing valve is therefore given by*

$$P_u = f(Q/\sqrt{P_A - P_u}) \quad \text{with} \quad P_A - P_u = \text{constant}$$

If the hydraulic forces are negligible, these characteristics become coincident, so that there would be no result for $P_u = f(Q)$ since there is a unique relationship between the displacement x and the quantity $Q/\sqrt{P_A - P_u}$ but none between the displacement x and the flow Q. Examples of this kind of characteristic are given in Chapter 4.

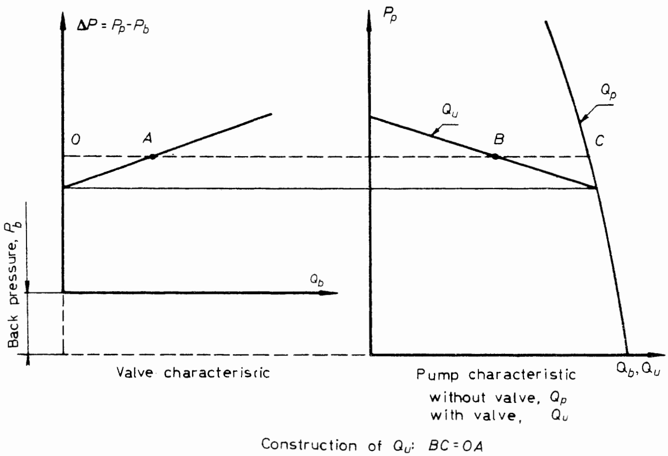


Figure 3.33

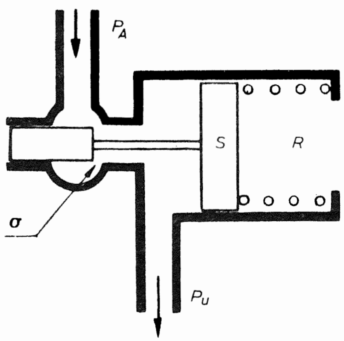


Figure 3.34

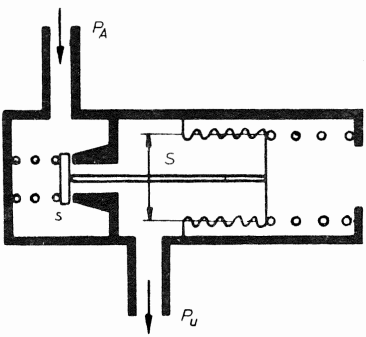


Figure 3.35

The sealed pressure-reducing valve—Sometimes the valve is required to keep the pressure P_u constant even for zero, or almost zero, flow. It is therefore necessary for it to be leakproof. Now, in practice it is almost impossible to achieve a perfect seal by a piston in a cylinder of equal size (cf. *Figure 3.34*). The seal is, however, obtained quite easily by the contact of two plane faces perpendicular to the direction of motion of the movable part.

In addition, this device reduces friction almost to zero, especially if the piston is replaced by a capsule. This is why many pressure-reducing valves, especially those designed for use with gases, have the basic form shown in *Figure 3.35*. The value of the pressure P_A directly affects the equilibrium, as can be seen in the figure. Thus, if

S = effective area of the capsule (cf. Section 4.2.3)
s = area of orifice
R = resultant stiffness of springs and capsule
x = displacement
F = resultant thrust from springs at $x = 0$
F_H = hydraulic force,

we have for equilibrium

$$(P_A - P_u)\,s + P_u S = F - Rx + F_H$$

If x is small (x and F_H negligible), this becomes

$$P_u = \frac{F}{(S - s)} - P_A \frac{s}{(S - s)}$$

In practice, for leakproof pressure-reducing valves, the characteristics used are

$$P_u = f(P_A) \quad \text{with} \quad \frac{Q}{\sqrt{P_A - P_u}} \text{ constant}$$

and for zero flow (*Figure 3.36*)

$$P_{u_0} = f(P_A)$$

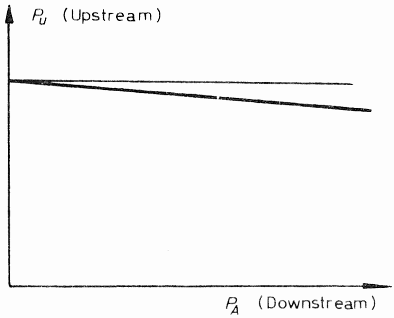


Figure 3.36

The slight increase in the downstream pressure, P_u, as the upstream pressure, P_A, is decreased is found in most commercial valves*. Some manufacturers have succeeded in overcoming this, but at the cost of an increase (sometimes troublesome) in the frictional forces opposing the force sP_A. The actual arrangement used is shown in *Figure 3.37*.

---

* To a certain extent, the quality of the valve can be assessed by the slope of the characteristic

$$p = \frac{\Delta P_u / P_u}{\Delta P_A / P_A}$$

The values of p may lie between $\frac{1}{20}$ and $\frac{1}{10}$. If s has been determined by the condition of maximum flow, the characteristic can be improved by increasing S.

Unloading (or by-pass) valve—The function of an unloading valve is to regulate the *flow* from a *flow generator*. A diagram showing its construction and operation is given in *Figure 3.38*. The usable flow, Q_u, is determined from the loss of head, ΔP, which occurs through the orifice, Σ. This loss of head is applied to the piston,

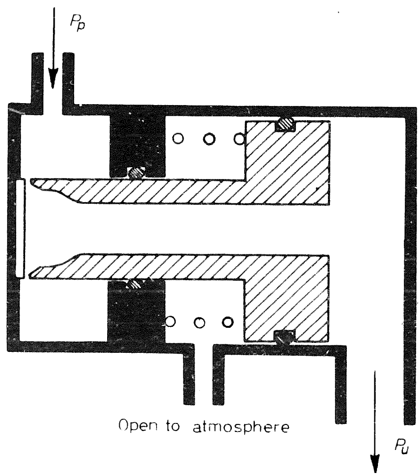


Figure 3.37

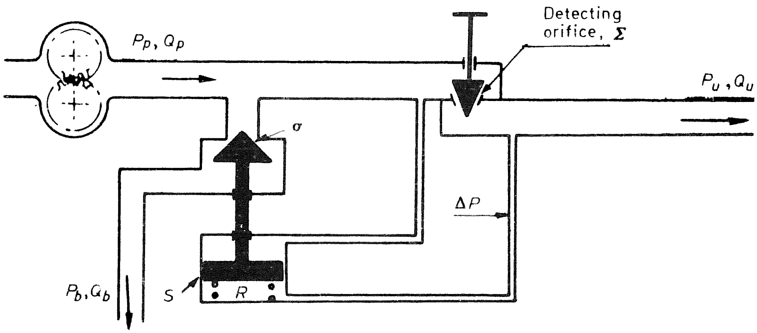


Figure 3.38

S, where it acts against the force F of the spring R. If the head loss is too high, the area σ is increased and more fluid flows into the by-pass circuit.

A perfect unloading valve maintains a constant value of Q_u. In fact, owing to the variation in the length of the spring with the opening of the valve and to the hydraulic forces developed from the by-pass flow, the usable flow, Q_u, is a function of the parameters $Q_b/\sqrt{P_p - P_b}$ (opening of the valve) and $P_p - P_b$ (loss of head across it).It can now be seen that it is impossible to define a characteristic of a pump with its regulator, since the pressure from the pump, $P_p = P_u + \Delta P$, is a function of the downstream characteristic of the circuit, $P_u = f(Q_u)$.

It is here that the main interest in this system of regulation of a positive displacement pump lies, as compared with the old system of a pressure-relief valve and throttling. For an unloading valve, with the loss of head, ΔP, low, the pump is required to produce just sufficient pressure for the circuit downstream.

For an unloading valve, the characteristics $\Delta P = f(Q_b)$ for $P_p - P_b$ constant, or P_p constant, are the most interesting, since they permit the determination of the operating point of a circuit for which the characteristics of the pump, downstream circuit and orifice are known. This determination provides a good example of the solving of a hydraulic problem with the use of characteristics and will thus be explained (*Figure 3.39*).

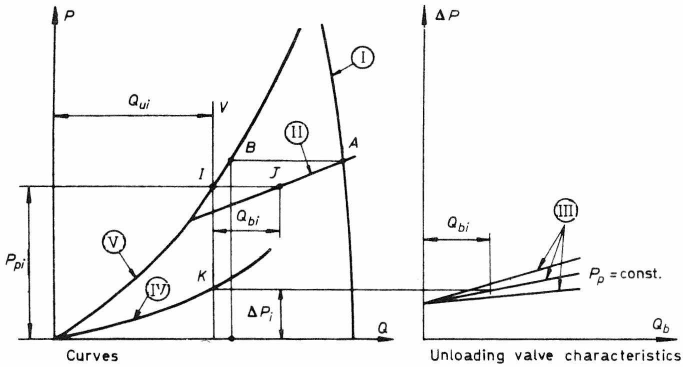


Figure 3.39

The operating point A required is at the intersection of the P_p, Q_p characteristic of the pump (curve I) and the P_p, Q_p characteristic of the aggregate of all the components situated downstream of the pump (curve II). In order to find it, we must plot the family of characteristics of the valve

$$\Delta P = f(Q_b) \quad \text{for} \quad P_p \text{ constant (curve III)}$$

and, for example, on the same graph
the characteristic of the pump: P_p, Q_p (curve I)
the characteristic of the orifice Σ: $P_p - P_u = \Delta P$, Q_u (curve IV)
the sum of the downstream characteristics + orifice Σ: $P_u + \Delta P = P_p$, Q_u (curve V).

To obtain any point J on curve II, start from the corresponding point I on curve V ($P_p = P_{pi}$, $Q_u = Q_{ui}$): at $P_p = P_{pi}$, we have $\Delta P = \Delta P_i$, the intersection K of the vertical V through I with curve IV. At $\Delta P = \Delta P_i$ and $P_p = P_{pi}$, we have $Q_b = Q_{bi}$ (on the curves III). Hence, point J is given by the ordinates P_{pi} and $Q = Q_{ui} + Q_{bi}$.

Curve II when plotted out cuts the characteristic curve for the pump at the operating point, A. The flow effectively used in the circuit is given by the Q ordinate of point B, which is the intersection of the horizontal through A with curve V.

Throttle valves—The function of a throttle valve is to regulate the flow from a *pressure generator*. Its layout is shown in *Figure 3.40*. In this case, the effect of

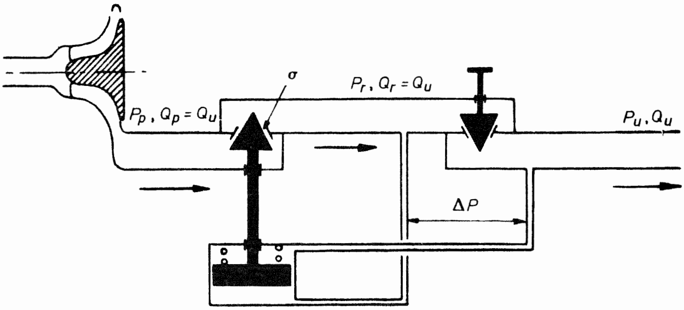


Figure 3.40

the loss of head, ΔP, is to throttle the flow in the main duct at σ. The theory is similar to that of the unloading valve.

The important characteristics are

$$\left.\begin{aligned}\Delta P &= f\left(\frac{Q_u}{\sqrt{P_p - P_r}}\right)\\ &= f(Q_u)\end{aligned}\right\} \quad \text{for} \quad P_p - P_r \text{ constant}$$

The pressure-relief, pressure-reducing, unloading and throttle valves described above are basic and of the simplest type. The working component is often merely controlled by the pressure (or pressure difference)* instead of being directly actuated by it as was shown here. This gives a better approximation to the theoretical characteristic, but there is the risk of the onset of certain instabilities.

Often, too, the fixed detection area, S, for unloading or throttle valves is not only variable but under the control of an external parameter. If, in addition, the spring, R, is replaced by a hydraulic force controlled by another external parameter, the regulated flow can be made equal to the *product* of *two independent functions.* (This is hydraulic multiplication.)

A large number of variations may be used, e.g. the flow regulator with the barometric correction potentiometer described in Section 3.4.2. But these elaborations do not affect the basic operation of the valves nor the fundamental diagrams given.

3.6. REDUCED CHARACTERISTICS

3.6.1. DEFINITION AND IMPORTANCE OF REDUCED CHARACTERISTICS

In the equation for the operation of a hydraulic component given in Section 3.1

$$f(\alpha, \beta, \gamma, \delta, \zeta, \eta, \theta \ldots) = 0 \tag{1}$$

we can observe or establish that certain variables are very often present only in the form of simple groups.

Thus, if there is a certain symmetry between two variables, only their difference, D, or ratio, R, can appear in the equation (see the example below, Section 3.6.2). Three variables can only appear in a homographic relation, H (see again Section 3.6.2). Often, considerations of dimensional analysis lead us to expect, and experiments confirm, that certain variables will be grouped in the form

$$A = \alpha\gamma^g \delta^d \zeta^z \quad ; \quad B = \beta\gamma^{g'} \delta^{d'} \zeta^{z'}$$

* For examples see e.g. Section 4.3.4.1.

The best example is Reynolds number, defined in Section 1.2.4; more particular examples are given in Sections 3.6.3 and 3.6.4.

It is therefore possible to decrease the number of variables in the equation of operation and to simplify it by replacing the initial variables, α, β, γ, δ, by the *reduced variables* D, R, H, A, B, . . .

Thus we have a *reduced equation of operation*

$$g(D, R, H, A, B, \ldots) = 0$$

and *reduced characteristics*, e.g.

$$g(D, A, R_i, H_j, B_k, \ldots) = 0$$

These characteristics are very important. A single reduced characteristic often completely defines a component and replaces several families of ordinary characteristics. Sometimes even the *dimensionless* combination of reduced variables, A/A_0, B/B_0, enables us to represent the whole category of components with a single characteristic, thus establishing laws of operation or rules of general use applicable to all components of this type (cf. Section 3.6.2).

Sometimes the grouping of certain variables in a single simple term, such as

$$A = a\gamma^g \delta^d \zeta^z$$

is only valid in a region Δ in the plane of the characteristic, outside of which the index of one of the variables, such as z of ζ, cannot be regarded as constant. So, rather than rejecting the reduced form, the characteristic will be plotted to give a unique curve inside the region Δ and will be divided outside it, each branch corresponding to a fixed value ζ_i of ζ.

The unusual appearance of these characteristics results in some fancy names: spike characteristic, horse tail characteristic, etc.; an example of the latter is given in Section 3.6.4.

3.6.2. EXAMPLE 1: THE HYDRAULIC POTENTIOMETER

The characteristic equation of the potentiometer, established in Section 3.4.2,

$$\frac{P - P_2}{P_1 - P_2} = \frac{1}{1 + \xi_1 S_2^2/\xi_2 S_1^2}$$

is a reduced characteristic between two variables

$$\pi = \frac{P - P_2}{P_1 - P_2} \qquad \sigma = \frac{S_2}{S_1}$$

the five original variables, P, P_1, P_2, S_1 and S_2, being replaced by the two new variables π and σ. These two variables are dimensionless, so that eqn. (4) represents the operation of all hydraulic potentiometers, whatever their size. It is of obvious advantage to be able to plot a single curve $\pi = f(\sigma)$ instead of an infinite number of curves of the type

$$f(P, S_2, S_{1_i}, P_{1_j}, P_{2_k}) = 0$$

The region of validity of the reduced characteristic is limited at low pressures only by the presence of laminar flow and at high pressures by the possible deformation in shape of the component or by compressibility effects in the fluid. The reduced characteristic is therefore valid for the whole normal operating region of the equipment.

3.6.3. EXAMPLE 2: THE CENTRIFUGAL PUMP

In Section 3.3.2.2 the problem of centrifugal pumps (*Figure 3.8*) was mentioned, with P, Q characteristics for constant speed of rotation, N, and constant inlet pressure, P_i. As can be seen from the geometry (*Figure 3.41*), the relative

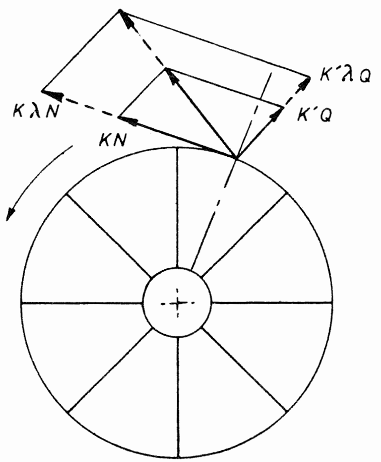


Figure 3.41

angles between each streamline of the fluid and the vanes of the pump (fixed and movable) are not changed if we multiply both the rotational speed, N (and therefore the tangential velocity of each point on the rotor) and the volume flow, Q, of the liquid (and therefore the velocity of flow) by λ. The flow pattern *remains the same.*

In addition, the intensity of the centrifugal action which causes the increase in pressure is multiplied by λ^2. It is therefore permissible to represent the operation of the pump independently from the velocity of rotation by the reduced coordinates Q/N and P/N^2.

In practice, the experimental values of Q/N and P/N^2 align the reduced characteristic very nicely on a single curve over a large range of speeds.

Points at high volume flow and low pressure sometimes lie off the curve, owing to cavitation. It is useful to give the limit for efficient operation of the pump in the form of a minimum value for the inlet pressure as a function of the flow (or speed of rotation). This would enable cavitation to be avoided and render the single reduced characteristic perfectly valid*.

* See also Chapter 10, Example 3.1.

3.6.4. EXAMPLE 3: FUEL CONSUMPTION OF A JET ENGINE

In all the problems of fuel supply and regulation for jet engines, it is necessary to know the fuel consumption as a function of certain parameters:

the speed of rotation, N

the pressure and temperature at the intake of the jet, p_1 and T_1

the ambient static pressure, p_0

and also certain size parameters such as the cross-sectional area of the duct, S_5.

Dimensional analysis considerations which have been verified by experience over a fairly wide range, where the effect of Reynolds number is negligible and the combustion efficiency is constant, show that the reduced characteristic equation has the form

$$\frac{Q}{p_1\sqrt{T_1}} = f\left(\frac{N}{\sqrt{T_1}}\right) \qquad \text{for sonic flow at the outlet of the jet}$$

$$\frac{Q}{p_1\sqrt{T_1}} = f\left(\frac{N}{\sqrt{T_1}}, \frac{p_1}{p_0}\right) \qquad \text{for subsonic flow at the outlet}$$

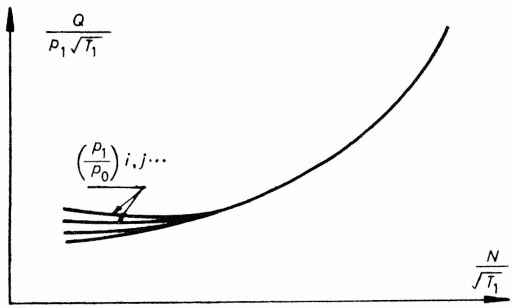


Figure 3.42

The reduced characteristic is plotted on $N/\sqrt{T_1}$ and $Q/p_1\sqrt{T_1}$ coordinates and has a 'horse tail' (*Figure 3.42*) in p_1/p_0 for low values of $N/\sqrt{T_1}$.

3.7. THE USE OF HYDRAULIC CHARACTERISTICS IN THE ANALYSIS OF SERVO SYSTEMS

The considerations given below concern servo systems, hydraulic or otherwise, in which components with non-linear characteristics are used. Their particular importance in hydraulics arises from the fact that, in this field, non-linearity of characteristics is the rule rather than the exception.

3.7.1. DETERMINATION OF THE STEADY-STATE CONDITIONS: t/s AND s/t CHARACTERISTICS

Consider the classical diagram of a servo system (*Figure 3.43*). An adder, Δ, compares the input, e, with the output, s, finds the magnitude of the error, $\epsilon = e - s$, and sends it to a component A which establishes the output according

to $s = f(\epsilon)$. For the analysis of a servo system we should consider separately
the adder, $\Delta[\epsilon = f(e, s) \bumpeq e-s]$
the component A, $[s = f(\epsilon)]$
and the total operation of the closed loop, $[s = f(e)]$.

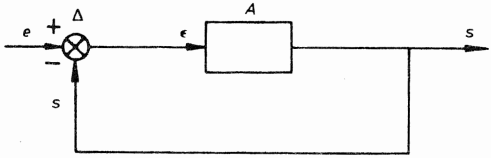


Figure 3.43

But it is sometimes difficult or even impossible to test the adder Δ and the component A separately. In this case, we will have to examine these two components together and replace the output, s, by applying a signal, s_1, to the adder (*Figure 3.44*); s_1 has the same nature as s but is independent from it.

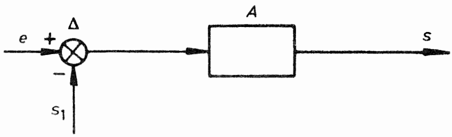


Figure 3.44

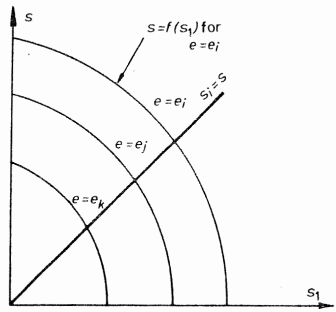


Figure 3.45

The result of this test can be expressed by the equation $s = f(s_1, e)$. By considering only the steady-state conditions, we can plot a *family of characteristics*, $s = f(s_1)$, for different values, e_i, of e (*Figure 3.45*). *Thus the operating point of a closed loop, in a steady-state condition, for input $e = e_i$, will be given by the intersection of the corresponding characteristic with the 45° line where $s_1 = s$.*

There is no assumption made of the linearity of A or Δ. This method of determining the steady-state operating point thus *takes into account any non-linearities* in the components of the servo system.

Note—In the straightforward case where the two components, Δ and A, are linear:

$$\epsilon = e_i - s_1$$

$$s = K\epsilon = K(e_i - s_1) \text{ (steady-state conditions)}$$

the family of characteristics reduces to a family of parallel straight lines of slope $-K$

$$s = K(e_i - s_1)$$

cutting the axes at points $s_1 = e_i$ and $s = Ke_i$.

We can see from *Figure 3.46* that the values of the error ϵ_i and the output s_i corresponding to the input e_i are given by the equations

$$\frac{s_i}{e_i} = \frac{K}{1 + K} \qquad \frac{\varepsilon_i}{e_i} = \frac{1}{1 + K}$$

which are fundamental equations for linear servo systems.

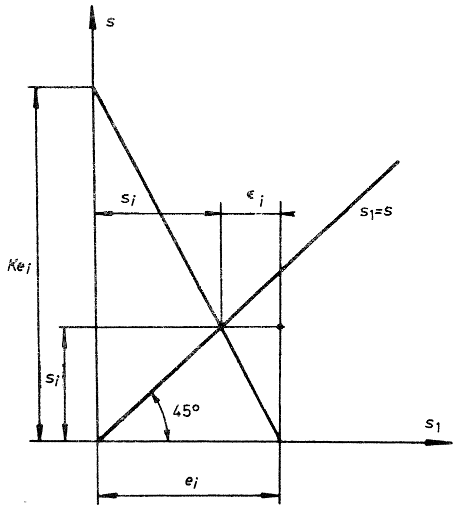


Figure 3.46

t/s and s/t characteristics—In practice, the component A usually consists of a number of components, A_i, A_j, . . . It is advantageous to group them into two chains, A_1 and A_2, either because precision or regulating components (chain A_1) are followed by power components (chain A_2) or because the open servo chain is made up of two successive items, or two groups of items, completely individual and easily separable. It is rare that the distinction between chains A_1 and A_2 cannot be made very easily. The block diagram is shown in *Figure 3.47*.

By introducing the intermediate variable, t, for the output from A_1 and the input to A_2, we can consider

(a) the family of characteristics of the chain $\Delta + A_1$, $t = f(s)$, for different values e_i of e; these characteristics are called t/s;

(b) the characteristic of the chain A_2, $s = f(t)$, referred to as s/t.

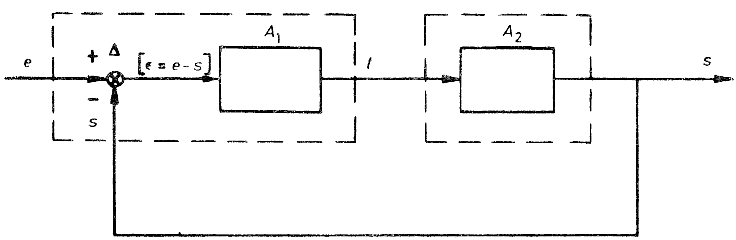


Figure 3.47

Closing the loop means giving chain A_2 the output t from chain $\Delta+A_1$ and feeding the output s from A_2 into the chain $\Delta+A_1$. *Therefore, the steady-state operating point of a closed loop for an input* $e = e_i$ *is given by the intersection of the* t/s *characteristic and the inverse of the* s/t *characteristic, plotted on the same graph* (curves C_1 and C_2, *Figure 3.48*). This method also takes into account any non-linearity of the characteristics. It is very useful, since it allows us to examine the t/s and s/t characteristics individually, as will be seen in the example given later.

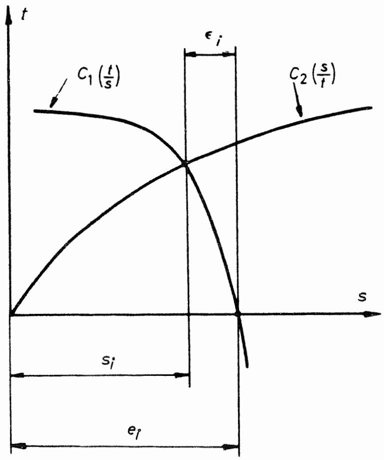


Figure 3.48

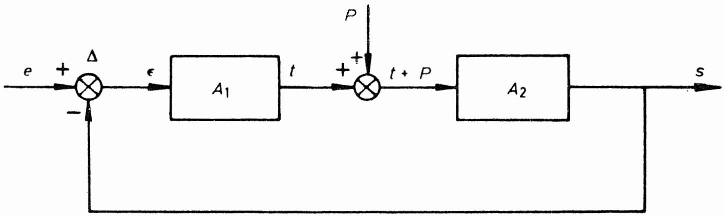


Figure 3.49

Figure 3.48 shows that the error, ϵ_i, tends to zero either when the slope of C_1 tends to infinity or when that of C_2 tends to zero. (If the slope of C_1 tends to infinity, it indicates that the gain of A_1 does so likewise; if the slope of C_2 tends to zero, it indicates that the gain of A_2 tends to infinity.)

The result that the static error is zero when there is an integration in the open loop (see Section 6.8.2), is simply a generalization of the known law for linear systems.

The introduction of a perturbation, P, between A_1 and A_2 means that t is replaced by $t+P$ as the input to A_2 (*Figure 3.49*). This results in an overall movement of the s/t characteristic by a distance P parallel to Ot (change of variable from t to $t+P$).

Figure 3.50 shows that the error ϵ_p *due to a perturbation* tends to zero as the slope of t/s at the operating point tends to infinity. This result is also an extension of the known theory of linear systems: a perturbation does not introduce an error into the output of a linear servo system, if the open loop has at least one integration upstream of the point where the perturbation is introduced (see Section 6.8.3).

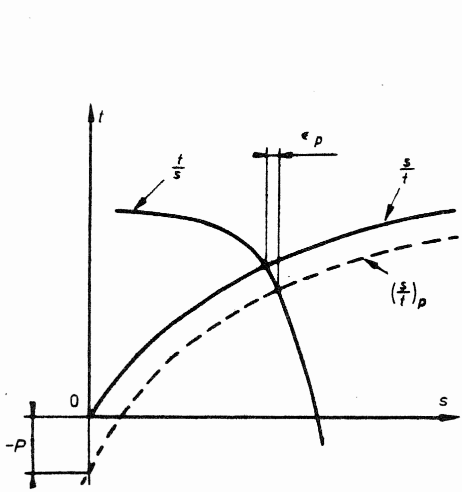


Figure 3.50

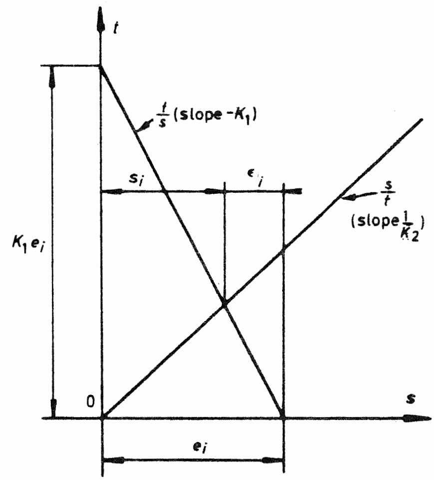


Figure 3.51

Note—In the straightforward case where all three components, Δ, A_1 and A_2, are linear

$$\varepsilon = e_i - s$$

$$t = K_1 \varepsilon = K_1 (e_i - s) \qquad \text{(steady-state conditions)}$$

and

$$s = K_2 t \qquad \text{(steady-state conditions)}$$

The t/s characteristics are the straight lines: $t = K_1(e_i - s)$, each one passing through the points ($t = 0$, $s = e_i$) and ($s = 0$, $t = K_1 e_i$) and of slope $-K_1$.

The s/t characteristic is the straight line $s/t = K_2$, which passes through the origin with slope $1/K_2$ on the t,s diagram (*Figure 3.51*).

The output, s_i, is the s ordinate of the point of intersection of these two straight lines and can be found by eliminating t between the two equations, from which we have

$$s_i = e_i \frac{K_1 K_2}{1 + K_1 K_2}$$

and

$$\varepsilon_i = e_i - s_i = \frac{1}{1 + K_1 K_2}$$

When a perturbation is introduced between A_1 and A_2, the s/t characteristic becomes $s = K_2(t+P)$, and the elimination of t gives the equations

$$s_i = e_i \frac{K_1 K_2}{1 + K_1 K_2} + P \frac{K_2}{1 + K_1 K_2}$$

$$\varepsilon_i = e_i - s_i = e_i \frac{1}{1 + K_1 K_2} - P \frac{K_2}{1 + K_1 K_2}$$

3.7.2. EXAMPLE: TACHOMETRIC REGULATION OF A JET ENGINE

In order to control the speed of rotation, N, of a jet engine, a regulator may be used to provide the necessary flow of fuel. The control signal for this regulator is the difference between the speed of rotation, N_0, selected by the pilot and the actual speed, N, of the engine.

The engine and the regulator are two separate items. The function of the former is to provide power and of the latter to regulate. There is therefore no difficulty in selecting the intermediate variable, t, which will be the flow, Q, of fuel from the regulator to the engine. The block diagram is shown in *Figure 3.52*.

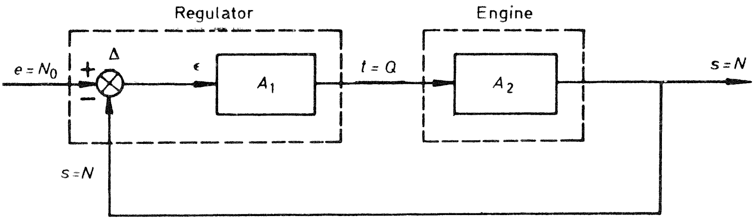


Figure 3.52

The family of t/s characteristics is here that of $Q = f(N)$ characteristics for the regulator for different values of the position of the pilot's manual control, $N_0 = N_{0i}$, and the s/t characteristic is the characteristic $N = f(Q)$ of the engine.

Consideration of the characteristics $Q = f(N)$ and $N = f(Q)$ has a number of advantages:

(1) The $Q = f(N)$ characteristic of the regulator can be determined while it is driven by a variable speed motor *independent of the engine.*

(2) The $N = f(Q)$ characteristic of the engine can be determined while it is supplied through a valve or any other regulator *independent of the regulator under consideration.*

(3) It will be shown below that this characteristic depends on a number of external parameters, the most important of which are

p_1, T_1 = pressure and temperature at the jet intake
M = flight Mach number

and on certain size parameters of the engine such as S_5, the cross-sectional area of the exhaust duct.

Having plotted the family of characteristics of the engine for different values of these parameters, we can plot the regulator characteristics on the same axes and find the operating points by reading off directly for the complete assembly of the jet engine and regulator together.

(4) Finally, note that it is usually unsatisfactory to separate the adder from the regulator, of which it forms an integral part.

The error signal is very often* made up of the difference between the following two forces, which are applied simultaneously to the spool of a control valve or to a servo motor controlling this spool: (*a*) the tension of a spring attached to the pilot's manual control (input $e = N_0$); (*b*) the centrifugal force developed by a movable part rotating at a speed proportional to the rotational speed of the engine (feedback signal, $s = N$).

3.7.3. PHYSICAL INTERPRETATION OF INTEGRAL CONTROL

The adjustment of a servo system is generally made on the precision chain, A_1. In Section 3.7.1 we saw that accuracy increased with the slope of the t/s characteristics. On the other hand, increase of slope generally has a detrimental effect on the stability of a system. The usual compromise must be made between accuracy and stability.

An excellent solution is provided by having a t/s characteristic with a small slope (good stability) which is displaced by a compensator with very slow action, in such a way as to nullify the error when the operating point itself is displaced (under the effect of a perturbation, for example).

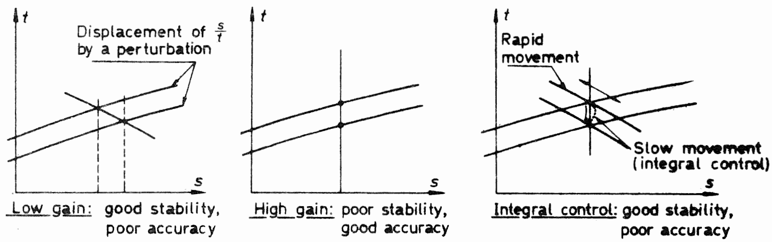


Figure 3.53

We can therefore distinguish two t/s characteristics, one having a small slope useful for rapid movements, thus countering instability, and the other having a high slope, so that each point is obtained only after a relatively long period of stabilization. The latter governs the accuracy of the system (*Figure 3.53*). All systems which include this second characteristic are called systems of integral control.

3.7.4. EXAMPLE: TACHOMETRIC REGULATOR WITH INTEGRAL CONTROL

The arrangement here described was, to the best of my knowledge, used for the first time by the British Lucas Company. The system supplying fuel to the

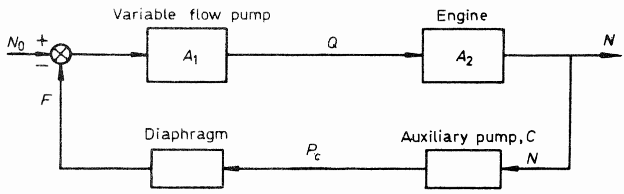


Figure 3.54

* In some regulators the error is of a different nature, electrical for example, N_0 and N being provided by a display potentiometer and tachometric generator, respectively. The measurement of ϵ and the assessment of the individual performances of adder and regulator proper are thus made easier, but nevertheless the Q/N and N/Q characteristics are still of interest.

engine consists of a variable flow pump. The variations of flow are controlled by the difference between two forces: (*a*) the tension in a spring (from the displayed velocity, N_0) and (*b*) the force F exerted on a diaphragm by the delivery pressure P_c from a small auxiliary centrifugal pump, C, which acts as a tachometric generator (to detect the actual speed, N). The block diagram, without integral control, is shown in *Figure 3.54*.

The gain of A_1 is low, so that the regulator is stable but lacks accuracy. It is therefore connected to the following system of integral control. The pressure, P_Q, of the fuel at the delivery from the variable flow pump feeding the jet engine (a pressure related in a unique manner to Q by the downstream impedance of the fuel circuit, i.e. in practice by the characteristic of the injectors) acts on the spring-loaded piston O, thus displacing a needle which is connected to a large dash-pot. The needle controls the area of passage between the delivery pressure of the main pump and that of the centrifugal pump acting as a tachometric detector. The pressure applied to the diaphragm can be regarded as the sum of two terms: P_c, the delivery pressure from the centrifugal pump rotating at

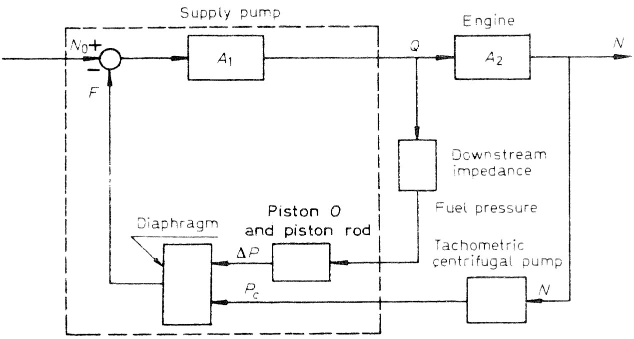


Figure 3.55

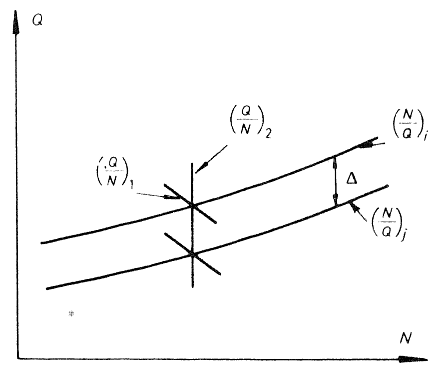


Figure 3.56

speed N when it is isolated from the delivery pressure of the main pump, and ΔP, the increase in P_c caused by the movement of the needle. ΔP is a function of P_Q, and therefore of Q, and follows the variation of Q with a large time lag, due to the dash-pot.

In this way, for rapid variations of the different variables there is no change in the operation without the integral control system: good stability remains good (*Figure 3.55*).

But for normal operation, the term ΔP is brought into action; if the needle is correctly arranged, this action rectifies the static error. The requirement is just that the gain of the *internal closed loop* (enclosed in the dotted line of *Figure 3.55*) should be large.

Now the two t/s characteristics (here, Q/N) introduced in Section 3.7.3 can be identified (*Figure 3.56*):
(1) the $(Q/N)_1$ characteristic, having a small slope for consideration of stability and withdrawn either by rapid measures made immediately after a variation of N, or by mechanically fixing the piston O;
(2) the $(Q/N)_2$ characteristic, having a high slope for steady conditions and withdrawn after stabilization of the piston O.

APPENDIX 3.1

EXAMPLES OF THE USE OF HYDRAULIC CHARACTERISTICS

A certain number of examples have been given in each Section. We will now include a classical example of the estimation of the performance of a hydraulic circuit and a specific example of the use of a reduced characteristic.

1. DETERMINATION OF THE MAXIMUM OPERATING TIME OF A RETRACTABLE UNDERCARRIAGE

The pressure of the accumulator is known: P_A. We must estimate the maximum force on the undercarriage (aerodynamic and gravitational), F_A; the maximum frictional forces on the ram, F_F; the minimum temperature of the oil in the accumulator.

The minimum pressure difference available for supplying the ram, ΔP_s, the difference between the pressure of the accumulator and the working pressure can be found from P_A, F_A and F_F. The maximum viscosity of the liquid and the characteristics of each component can be found from the minimum temperature.

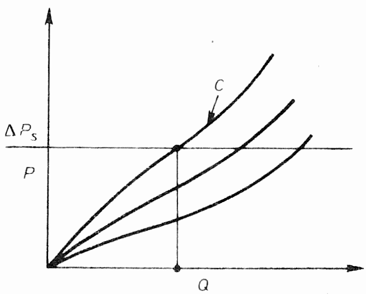


Figure 3.57

The addition of these elementary characteristics gives the characteristic of the complete line between the accumulator and the ram, C (*Figure 3.57*). The intersection of C with ΔP_s indicates the flow in the line from which the time taken to fill the ram can be calculated.

2. PROTECTION OF A JET ENGINE FROM HUNTING

This example is included to show how the use of reduced characteristics clarifies and simplifies certain problems. Hunting is a dangerous phenomenon in the mechanical behaviour of jet engines which starts when the flow of air through the compressor falls below a certain value. Since we are not concerned with the details of this feature, we will just accept that a curve representing its onset can be drawn on the graph of $M\sqrt{T_1}/p_1$ against $N/\sqrt{T_1}$, these coordinates being the reduced mass flow of air and the reduced rotational speed. In a similar manner, this curve (eventually with a horse tail) can be drawn on the graph of $Q/p_1\sqrt{T_1}$ (where Q is the flow of fuel) and $N/\sqrt{T_1}$ since, for a fixed geometrical shape, the air flow depends only on $N/\sqrt{T_1}$, on $Q/p_1\sqrt{T_1}$ and eventually on the ratio p_1/p_0 if the air flow at the jet exit is subsonic.

When a rapid acceleration is required from a jet engine rotating at speed N_i, it is necessary to provide it instantaneously with the maximum flow of fuel.

Thus we can appreciate the necessity for a device to limit the supply of fuel to a level a little lower than the danger level for all values of the displacement and speed of displacement of the manual control and for all values of the external parameters, p_1 and T_1.

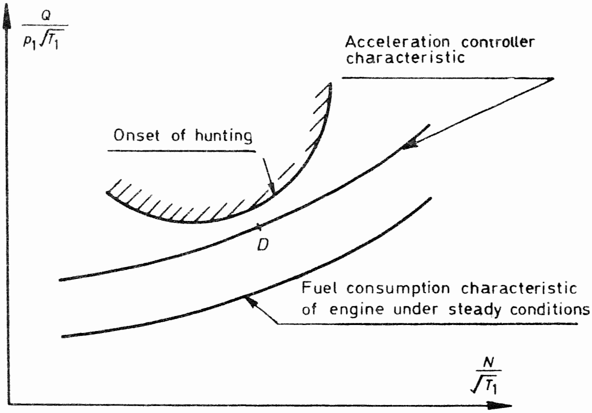


Figure 3.58

This device is called an acceleration controller. It fixes a limiting value of $Q/p_1\sqrt{T_1}$ as a function of $N/\sqrt{T_1}$, i.e. the ideal characteristic (*Figure 3.58*). The design of a controller rigorously following this relationship, in which f is not a well defined mathematical function, would be very difficult. An approximate function, for example $Q = p_1 f(N)$, is therefore often accepted, although this may be too approximate if the variations of T_1 are significant.

The following principle for establishing a similar relationship is more suitable. It is the same as the ideal curve at the most dangerous point, D (cf. *Figure 3.58*).

At point D, the ideal characteristic is very similar to the tangential curve*

$$Q/p_1\sqrt{T_1} = K(N/\sqrt{T_1})^{\alpha}$$

which can be written

$$Q = K\sqrt{T_1}\,p_1(N/\sqrt{T_1})^{\alpha} = K\,N\,p_1(N/\sqrt{T_1})^{\alpha-1}$$

Hence it is easy to attain $Q = K\,.\,N\,.\,p$ where p is some pressure.

But the characteristic of the compressor in the vicinity of the reduced rotational speed at point D, $(N/\sqrt{T_1})_D$, is similar to the tangent characteristic†

$$p_2/p_1 = (N/\sqrt{T_1})^{\beta}$$

Now, for the corresponding ratio p_2/p_1, the characteristic of the pneumatic potentiometer (*Figure 3.59*) is similar† to

$$p'/p_1 = (p_2/p_1)^{\mu}$$

* This means that, when the ideal characteristic is plotted on logarithmic coordinates, it approximates to a straight line L over a reasonably large region. This similarity to a logarithmic straight line is closer than to a straight line on linear coordinates and is more useful for the estimation of variations (see Chapter 5). The coefficient α is the slope of the logarithmic straight line, L.

† The coefficients β and μ are found in the same way as α.

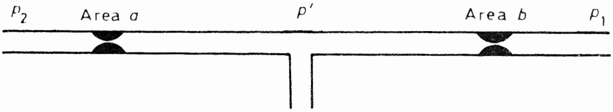


Figure 3.59

where μ is a function of the ratio a/b of the cross-sectional areas of the inlet and outlet orifices of the potentiometer (cf. Section 3.4.2).

The result is that

$$p'/p_1 = (p_2/p_1)^{\mu} = (N/\sqrt{T_1})^{\mu\beta}$$

and if a/b, and thus μ, is selected in such a way that $\mu\beta = \alpha - 1$, the pressure p', applied to a simple apparatus obeying the relation $Q = K . N . p$, gives a flow

$$Q = K N p' = K N p_1 (N/\sqrt{T_1})^{\alpha-1}$$

i.e. the *required flow*.

The necessary value of μ can be achieved in practice: the principle holds and the equipment is found to give satisfactory results.

REFERENCE

[1] Faisandier, J. *Les Mécanismes Hydrauliques*, p. 85 *et seq.*, Paris, Dunod, 1957

4

FORCES IN HYDRAULIC SYSTEMS

4.1. NATURE OF THE FORCES CONSIDERED

It is not necessary to consider forces whose origin is mechanical, electrical, etc. which are encountered in most equipment, hydraulic or otherwise. We shall only consider forces which are either *truly hydraulic*, i.e. those which arise from the action of a fluid on a surface, or *related indirectly* to the presence of liquid in the equipment under consideration, such as the forces developed in components with seals, frictional forces in seals, elastic forces in bellows and capsules, etc.

True hydraulic forces can always be effectively expressed in the form of an integral $\int P\,dS$ over the surface S. It is, however, convenient to differentiate between the forces exerted by a fluid which may be at rest (for want of a better name, these are called *static forces*) and the forces due to the flow itself, which are called *dynamic forces*.

This Chapter will therefore deal with static hydraulic forces, dynamic hydraulic forces and forces due to seals. In conclusion some general considerations relative to force and stiffness will be added.

4.2. 'STATIC' HYDRAULIC FORCES

4.2.1. HYDRAULIC FORCES EXERTED ON A PISTON

The simplest example of hydraulic force is that of the force, F, developed on a piston of surface area S by a fluid at pressure P

$$\boxed{F = SP} \tag{1}$$

The calculation of such a force is quite straightforward. It is important, however, that all the piston forces be included when considering the total action on a composite piston. This can be simplified by considering it as being made up of a number of *independent cylindrical pistons* and finding the resultant force applied to each:

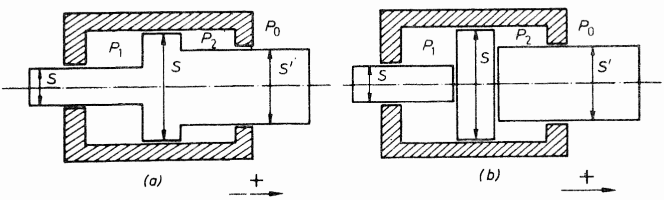


Figure 4.1

$$(a)\ \Sigma F = P_0 s + P_1 (S - s) - P_2 (S - S') - P_0 S'$$
$$(b)\ \Sigma F = s (P_0 - P_1) + S (P_1 - P_2) + S' (P_2 - P_0)$$

In *Figures 4.1a* and *b*, the forces exerted on a three-stage piston have been expressed by the usual method and by that of dividing it into basic cylindrical pistons, respectively.

4.2.2. OTHER 'STATIC' HYDRAULIC FORCES

There are other 'static' hydraulic forces which are more difficult to calculate. These are the cases where either the surface area, S, is not immediately known (e.g. bellows, diaphragms or capsules) or where the pressure, P, is not constant, as in the case where the surface of contact is perpendicular to the force considered and separates two chambers at different pressures, or for valve plates and certain seals, valves and fluid transportation devices, etc.

4.2.3. DETERMINATION OF THE EFFECTIVE SURFACE AREA OF CAPSULES, DIAPHRAGMS, BELLOWS, ETC.

4.2.3.1. *Type of component*

The type of component considered is one which has a deformable fluid-tight surface placed between two regions of fluid at different pressures, such as capsules, diaphragms and bellows, as shown diagrammatically in *Figure 4.2*.

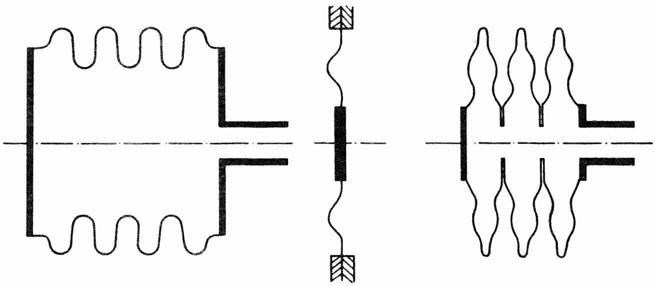


Figure 4.2

Components of this type with metal bellows have a significant mechanical stiffness. Rubber or plastic bellows, on the other hand, have negligible stiffness and are sometimes artificially stiffened by the addition of one or more springs.

4.2.3.2. *Concept of effective surface and rate. Numerical valves*

Consider a hydraulic component of the type described, a capsule for example. An increase of internal pressure, ΔP, tends to extend the capsule by a length Δx. This extension can be cancelled out by the application of a force, ΔF, along the axis of the capsule (*Figure 4.3*). It is therefore natural to define the *effective surface* of the capsule as

$$S_e = \frac{\Delta F}{\Delta P} \tag{2}$$

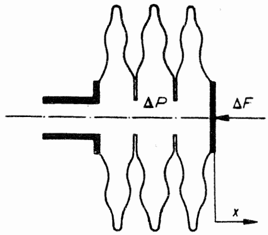


Figure 4.3

i.e. the area of piston which would be kept in equilibrium by a force ΔF when acted on by a pressure ΔP. Similarly, the *rate* of the capsule is given by

$$A = \frac{\Delta x}{\Delta P} \tag{3}$$

i.e. the ratio of the increase in length to that of pressure while under constant mechanical load, F_0. The rate has the dimensions $F^{-1}L^3$. It is related to the *mechanical stiffness*, R, and the effective surface area, S_e, of the capsule by the equation

$$A = \frac{S_e}{R} \tag{4}$$

It should be noted that the rate of a capsule is not necessarily constant. It will vary with load and position.

Note—The change in pressure, ΔP, should be considered as an algebraic quantity. For mechanical reasons it is often necessary to use capsules where the excess pressure is applied externally. For diaphragms and membranes, the sign of ΔP obviously depends only on the choice of the positive axis.

The methods of using capsules, membranes, diaphragms, etc. can vary considerably. In practice, however, one or other of the following methods is adopted:

(1) keeping the *length constant* or quasi-constant; the capsule therefore gives a *force* ($S_e \,.\, \Delta P$);

(2) keeping the *force constant*; the capsule therefore gives a *position*.

In the first case, the important factor is the *effective surface*. The slope does not affect the operation at constant length. In the second case, the important parameter is the rate. It is therefore the *effective surface area* and *slope* which determine the tolerances in manufacture necessary for good operation of the equipment. The two parameters, S_e and A, are usually quoted by capsule manufacturers.

The ratio S_e/S_t, of effective area to total area, is a very useful parameter for dimensioning in the initial design stage. Other important factors are the pressure limit and force limit for the capsule and also the operating temperature.

The characteristics of standard capsules supplied by the Callisto Company* are given in *Figure 4.4*.

* Société Callisto, 96 Avenue du General-de-Gaulle, La Garenne-Colombes (Seine).

External diameter, mm	*Total area,* S_t, cm^2	*Effective area**, S_e, cm^2	*Ratio* S_e/S_t	*Diaphragm thickness,* mm	*Rate,* mm/ kg/cm^2	*Maximum* ΔP, kg/cm^2	*Alloy used*
20	3·14	0·8	0·255	0·07	1	1	UBe2
				0·1	0·4	2·5	
				0·3	0·01	25	Refractory, e.g. Nimonic, ATGS3, etc.
				0·4	0·002	60	
				0·6	0·002	100	
				0·8	0·001	300	
30	7·08	2·7	0·381	0·1	1	1	UBe2
				0·2	0·30	2	
				0·2	0·125	8	Refractory
				0·3	0·06	12	
				0·4	0·03	25	
				0·6	0·016	50	
				0·8	0·009	80	
45	15·9	5·8	0·365	0·1	3·5	0·6	UBe2
				0·2	1·2	1	
				0·15	1	2·5	Refractory
				0·2	0·6	2·5	
49	18·9	6	0·317	0·2	1	2	Refractory
				0·3	0·4	5	
				0·4	0·2	10	
				0·6	0·1	20	
54	22·9	7·5	0·327	0·4	0·2	10	Refractory
				0·6	0·13	15	
56	24·7		0·304	0·1	2·5	1·1	UBe2
				0·2	1·8	1·1	
				0·35	1	4	
65	33·2	12	0·36	0·15	3·6	0·5	Refractory
				0·20	2·8	0·8	
				0·07	13·5	0·1	UBe2
				0·10	10	0·2	
				0·20	3·4	0·4	
76	45·2	20	0·440	0·07	100	0·020	UBe2
				0·30	14	0·100	Refractory
100	78·8	30	0·38	0·07	100	0·01	UBe2
				0·10	70	0·03	

* The effective area may slightly differ from the values shown.

Figure 4.4. Characteristics of standard capsules (single element)

4.2.3.3. *Variations of the effective area and rate*

Suppose that the capsule is subjected to an initial pressure, P_0, with an initial force, F_0 and length, x_0. By applying a pressure *increase*, P, and measuring the increase of length, x, and the increase, F, of applied force necessary to return the capsule to its initial position, the values of S_e and A at the initial conditions, P_0, F_0, x_0, can be determined. The resulting values are often significantly different from those corresponding to $P_0 = 0$ and $F_0 = 0$.

Except in the case of capsules specially designed to maintain constant rate, significant variations in A can be regarded as normal. In fact, it is difficult to obtain a linear deformation of the diaphragm with pressure (or force). On the other hand, the variation of the effective area must be kept within rigid limits of the order of magnitude of the accuracy of the readings. There must be no snagging or contact between the sides of the capsule (discontinuities).

In the construction of an accurate instrument, a regulator for example, it is essential that the following values are known at the actual operating position: either S_e, i.e. operation at constant length and variable force, or A, i.e. operation at constant force and variable length, or S_e and A, i.e. operation with variable force and length. In practice, these values can be found directly from the characteristic curves of the capsule considered, according to the method of use proposed (*Figures 4.5* and *4.6*).

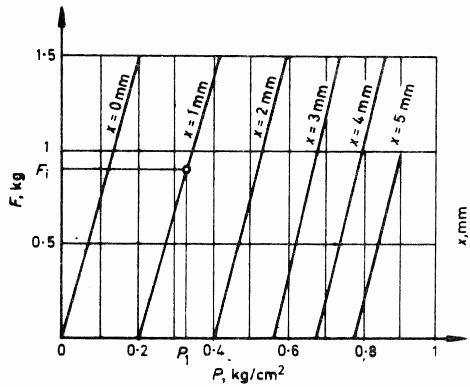


Figure 4.5

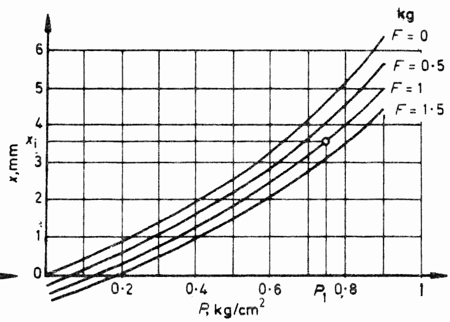


Figure 4.6

Figure 4.5 shows the variation of F with P at constant length (x = constant). The value of S_e at $P = P_i$ and $F = F_i$ is given by the slope of the curve for $x = x_i$ at that point. (Note that the curves for constant length are practically parallel straight lines, so that S_e remains almost constant over the operating range.)

Figure 4.6 shows the variation of x with P at constant force. The value of A at $P = P_i$ and $x = x_i$ is given by the slope of the curve for $F = F_i$ at that point. The figures, provided by the Callisto Company, represent a family of *real* characteristics for a capsule manufactured for aircraft instrument panels and specially designed to have a variable rate.

4.2.3.4. *Theoretical determination of effective area*

The theoretical determination of the effective surface area is obviously possible from a consideration of the shape and size of the capsule, knowing the mechanical properties of the metal used. For example, it is possible to determine $x = f(P)$ for $F = F_i$ and $x = f(F)$ for $P = P_i$, and hence

$$S_e = \frac{\mathrm{d}F/\mathrm{d}x}{\mathrm{d}P/\mathrm{d}x}$$

This method is seldom used, since the calculations are complex and not very accurate in a non-linear region.

By examining the deformation of the capsule, however, it is possible to estimate the effective surface area and, in particular, to predict the variations entailed by a modification in the shape of the folds.

Suppose the capsule consists of a single diaphragm, as shown diagrammatically in *Figure 4.7a* in its initial position. *Figure 4.7b* shows the diaphragm subjected to a pressure P; in *Figure 4.7c* it is subjected to a force F, and *Figure 4.7d* shows it simultaneously subjected to a pressure P and force F.

In the change from (a) to (b), the pressure P has done work, W_1. In the change from (b) to (d), the force F has done work, W_2. At (d), the diaphragm has almost returned to its initial shape (a), so that the energy stored in it is small, and $W_1 \simeq W_2$.

Estimation of W_2:

$$W_2 = \frac{F}{2} x$$

Estimation of W_1—Consider an element of area $\mathrm{d}S$ of the membrane. Under the action of the pressure, P, the displacement of this element is λx, where λ is a function of the position of the element on the diaphragm, normally between 0 and 1. Integrating over the surface gives

$$W_1 = \iint_{(S)} \frac{P}{2} \lambda \, \mathrm{d}S \, x = \frac{P}{2} x \iint_{(S)} \lambda \, \mathrm{d}S$$

If $W_1 = W_2$

$$S_e = \frac{F}{P} = \iint_{(S)} \lambda \, \mathrm{d}S$$

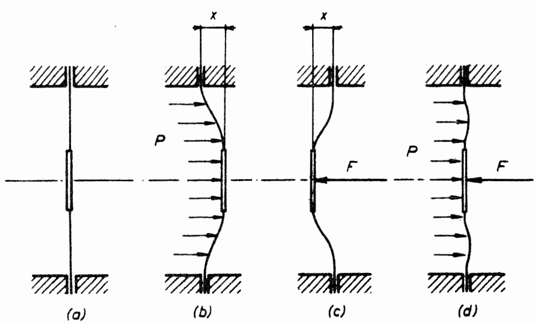


Figure 4.7

This means that, if x is the displacement of the centre of the diaphragm, and V the volume of fluid displaced when the capsule is subjected to a pressure P, then the *effective area* $= V/x$.

Therefore, in order to increase S_e, the diaphragm can be either stiffened in the central region (by increasing the area of the rigid part, for example) or the perimeter can be lengthened (by increasing the external diameter).

To a first approximation, the effective surface area is given by

$$S_e = \frac{\pi}{4}\left[\frac{D_0 + D_i}{2}\right]^2$$

where D_0 is the outside diameter (that of the casing) and D_i the inside diameter (that of the rigid central disc).

4.2.4. DETERMINATION OF THE APPLIED PRESSURE FOR VALVING SURFACES, SEALS, VALVES, ETC.

4.2.4.1. *Relevant components*—These are components in which two regions at different pressures are separated by the contact of two surfaces. The force exerted by the fluid obviously depends on the pressure distribution over the surfaces.

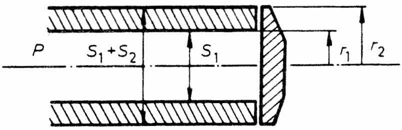


Figure 4.8

The simplest example of this type of component is the valve shown in *Figure 4.8* where the contact surfaces are plane annuli. Other, similar, components will also be considered. Another example is the development of radial hydraulic forces in a cylindrical valve which cause the well-known phenomenon of sticking. A simple analysis of valve sticking is given in *Appendix 4.2.*

4.2.4.2. *Theoretical evaluation and practical approach* (*for surfaces which are effectively planar*)

The hydraulic force exerted by the liquid at pressure P on the cap of this type of valve when it is *closed* (or almost closed)* is given by the sum of the force on the central part of the cap

$$F_1 = S_1 P$$

and the force F_2 on the annular region S_2. The latter depends on the decrease of pressure between r_1 and r_2, but if $p(r)$ is this pressure distribution, then

$$F_2 = \int_{r_1}^{r_2} 2\pi r p(r)\, dr \tag{9}$$

In specialized treatments of this subject, the estimation of $p(r)$ is made using differing theories depending on the nature of the flow (molecular, laminar or turbulent) and on the disposition of the surfaces (parallel, converging or diverging). These analyses are of little use, since the validity of the results

* When the valve is sufficiently open to allow an appreciable flow to take place, the acceleration of the fluid at the restriction affects the static pressures and therefore alters the forces exerted. When the valve is wide open, dynamic forces are present (cf. Section 4.3.2.1 and, in particular, the footnote on p. 109).

requires perfect surfaces and a knowledge of the deformation of the surfaces when under load, which is seldom achieved in practice.

In practice, it is preferable to reduce the area of S_2 to a reasonable minimum and to make an approximate evaluation of the force exerted on it (over- or underestimating according to whether the force required has an upper or lower limit). The force F_2 is expressed in the form

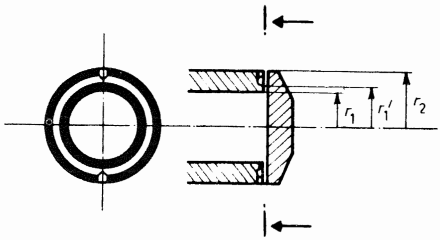
$$F_2 = \lambda S_2 P \tag{10}$$

where λ is a limit coefficient evaluated from experience.

The first difficulty encountered in the reduction of S_2 is the limit on the compression stresses in the material of the contact surfaces.

These surfaces cannot be hardened indefinitely. Another solution is to reduce the area S_2 without reducing the area of contact. An example of this is shown in *Figure 4.9*, where the contact surface is divided into two by an annular groove connected to one of the two pressure regions by a hole or communicating groove.

Figure 4.9



The region in which the pressure is unknown is reduced to the annulus between r_1' and r_1, while the area of contact is still the annulus between r_1 and r_2 (except for the area of the grooves).

The second difficulty in the reduction of S_2 is the weakness of the material. Finally there is the problem of making the valve fluid-tight—a problem which is not necessarily eased by increasing the contact area.

4.2.4.3. *Piston pump valve plate*

In a hydraulic piston pump, the liquid is distributed to the cylinders by a flat circular plate, placed perpendicularly to the axis of rotation, known as a valve plate or porting plate. This rubs against the rotor and has two slots, one associated with the low pressure and the other with the high pressure (*Figure 4.10*).

The liquid in these slots (in particular the delivery slot) together with any liquid present between the two surfaces in contact, exerts a *repelling force*, F_R, on the rotor.

On the other hand, there are a number of forces*, ΣF_P, which act in the opposite direction and tend to push the rotor against the valve plate.

The pump will not operate if $F_R > \Sigma F_P$.

* These forces are derived from the reaction of the hydraulic force exerted on the pistons. If there are n pistons, each of area s and at pressure P, $F = nsP$.

In fact, the clearance which is sufficient to allow complete leakage of the flow is so small that it is impossible to arrange a mechanical device which would efficiently limit this separation. On the other hand, due to problems of wear, overheating and possible seizing up, which become more severe as $\Sigma F_P - F_R$ increases, *correct operation may break down if* $\Sigma F_P \gg F_R$.

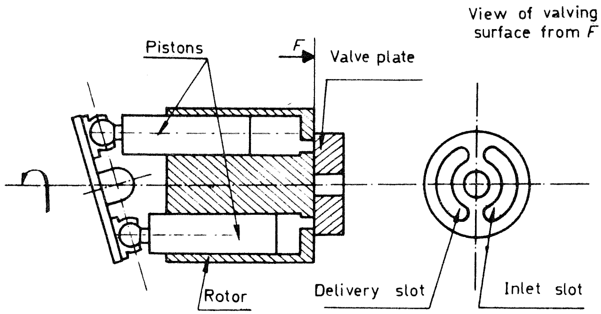


Figure 4.10

From a large number of tests, the following simple empirical rule has been deduced. Consider the contact surface, S, between the valve plate and the rotor (the shaded surface in *Figure 4.11*) bounded by the lines L_1, which represents the boundaries of the low-pressure (L.P.) region, and L_2 which bounds the high-pressure (H.P.) region. Draw the line L, equidistant from L_1 and L_2 (dashed); it encloses a surface area, S_e (hatched).

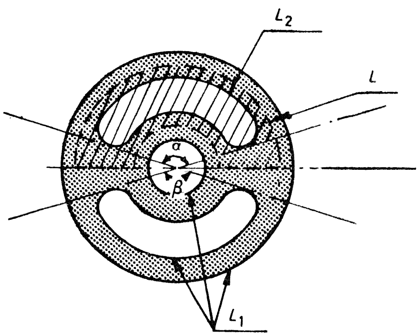


Figure 4.11

For a pump in which the inlet and outlet slots subtend equal angles at the centre ($\alpha = \beta$), the first condition necessary for operation* is

$$\frac{S_e}{ns/2} < 0{\cdot}95 \tag{11}$$

* This condition for contact is not sufficient in itself: the resultant ΣF_P must act within the contact surface, otherwise the distribution face would overturn.

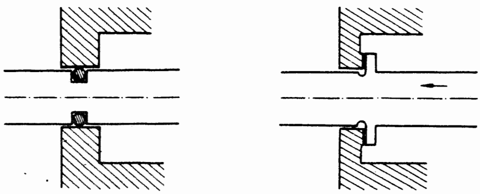
where n = number of pistons (odd or even), s = area of each piston. For the reasons of endurance given above, a lower limit is usually adopted, so that

$$0{\cdot}85 < \frac{S_e}{ns/2} < 0{\cdot}95 \tag{12}$$

4.2.4.4. *Seals rotating at high speed*

For seals rotating at high speed, it is essential to use materials which have very low coefficients of friction in order to avoid overheating, excessive wear or even complete seizing-up.

No material exists which has both the low coefficient of friction and the elasticity necessary for seals where the friction surface takes the form of a cylinder whose axis is coincident with the axis of rotation (*Figure 4.12a*). In order to make a sealed joint which can rotate at high speed, it is therefore necessary to use a friction surface lying in a plane perpendicular to the axis of rotation, together with a device for taking up any play (diagrammatically shown in *Figure 4.12b*).



(a) (b)

Figure 4.12

This type of layout obviously introduces balancing problems. If the pressure, P, is low, then neglecting the force of the spring (*Figure 4.13a*), which is small since the spring is included only to maintain contact when starting, the compression force acting between the stationary part, A, of the seal (usually made of pure carbon or copper carbon) and the movable part, B (usually made of Stellite steel)* is given by

$$F = \lambda(S_2 - S_1)P \tag{13}$$

where λ, a coefficient lying between 0 and 1, is a function of the pressure distribution between the radius r_2 of S_2 and r_1 of S_1 and, in practice, usually has the value $0{\cdot}5 \pm 0{\cdot}1$. The compression force is distributed over the surface contact area $S_2 - S_1$. The surface shear stress is therefore in the region of λP. For high pressures (above 50 kg/cm^2) this shear stress is prohibitively large.

The seal must therefore be balanced. This can be done by machining a shoulder on the shaft behind the sliding seal, C (*Figure 4.13b*), in order to make a seal between the shaft and the rotating part of the joint B. Force F becomes

$$F = [\lambda(S_2 - S_1) - (S_1' - S_1)]P \tag{14}$$

* The hydraulic reactions on the *shaft* are assumed restrained by a bearing (or a stop) not shown in *Figure 4.13*.

Obviously, too great a value of S_1' would change the sign of F and cause the contact surfaces to separate.

The balance coefficient, b, is defined as the ratio of the balance surface to the contact surface

$$b = \frac{S_1' - S_1}{S_2 - S_1} \tag{15}$$

The manufacturers* use a balance coefficient which ensures that separation of the surfaces does not take place. The value of this coefficient is approximately 0·3 for large seals, $(D_2-D_1)/2 \simeq 3$ mm, decreasing to about 0·15 for small ones, $(D_2-D_1)/2 \simeq 1$ mm.

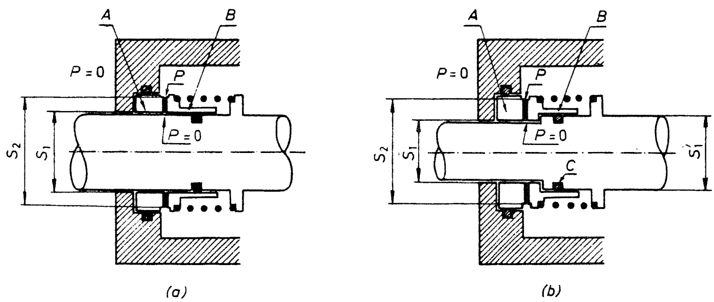


Figure 4.13

A balanced seal with full precautions against deformation of shape and fracture of the friction elements can operate under a pressure† of 150–200 kg/cm^2 and at speeds (peripheral speed at diameter D_2) up to 100 m/sec.

4.2.4.5. *Flapper-and-nozzle valves*

In the control of the variable orifices of hydraulic potentiometers (cf. Section 3.4.2) the energy level is often very low compared with the friction in a standard one-cylinder control valve (*Figure 4.14a*). The variation of flow area can be advantageously obtained by placing a flat plate perpendicular to the circular area of an orifice O. This arrangement is known as a flapper-and-nozzle valve.

Figure 4.14 is a diagrammatic sketch of a Lucas half-ball flapper-and-nozzle; both the half ball and the orifice are made of tungsten carbide. The hemispherical surface of the half ball allows self-righting of the flat surface, π, with a minimum of friction. For an orifice, O, of internal diameter 1 mm, the external diameter is 1·3 mm and the tops of the external projections are made absolutely coplanar with the exit of the orifice to avoid any rocking of the hemisphere.

In regulation problems it is useful to know the effective area, S_e, which is defined as the surface area of a small piston which, when subjected to the same

* Pacific seals, Sealol, etc.

† Different materials are sometimes used for very high pressures: bronze (with 8–12 per cent lead) instead of carbon for element A, nitrided steel instead of Stellite steel for element B.

pressure P, would develop a force equal to that exerted by the liquid on the hemisphere. A knowledge of S_e gives the force $F = S_eP$ (about 1 kg for the orifice described above when P is about 100 kg/cm^2) and, in particular, *the hydraulic stiffness of the flapper-and-nozzle valve in a given potentiometer*, $R_H = \mathrm{d}F/\mathrm{d}x$, which can be calculated from the relationship $P = f(x)$ of the valve.

In standard flapper-and-nozzle valves, the slope of $P = f(x)$ at the operating point is high, so that the values of $R_H = S_e\, \mathrm{d}P/\mathrm{d}x$ are very much greater than the stiffness of the metal springs of the valve.

The hydraulic stiffness of the potentiometer used as an example in Section 3.4.2, for which

$$\frac{\mathrm{d}P}{\mathrm{d}x} = \frac{(55-50)\ \mathrm{kg/cm^2}}{0{\cdot}0065\ \mathrm{mm}} = 7{\cdot}7\ \mathrm{kg/mm^3}$$

has values of $R_H = 6{\cdot}1$, $10{\cdot}2$ and $8{\cdot}0$ kg/mm, based on
the internal diameter, $D_i = 1{\cdot}0$ mm
the external diameter, $D_e = 1{\cdot}3$ mm
the mean diameter, $D_m = 1{\cdot}15$ mm

In addition, the actual areas S_e measured for different valves manufactured in a single batch are not consistent, so that it is preferable to avoid the use of this area as an operating parameter. This can be done by introducing, in parallel

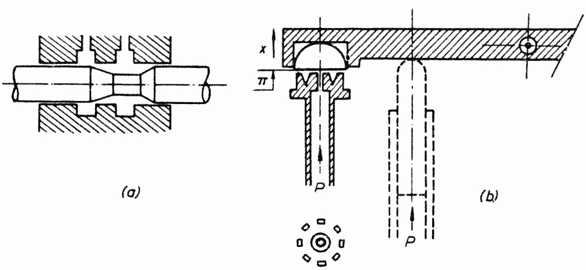


Figure 4.14

with R_H, another known constant hydraulic stiffness much larger than R_H (for example, by a small piston exposed to the pressure P), which acts on the flapper arm and whose cross-sectional area is greater than O_2 (as shown by the dashed lines in *Figure 4.14b*).

4.3. DYNAMIC HYDRAULIC FORCES

The determination of the dynamic hydraulic forces developed in components is an extremely difficult problem. To my knowledge, no satisfactory solution has been found, despite the efforts of a large number of researchers in every country.

This Section will be confined to expressing the problem clearly, to calculating the order of magnitude of the forces in question and to describing a proved empirical method of compensating these.

4.3.1. THE ORIGIN OF DYNAMIC FORCES

When a homogeneous fluid jet, of mass flow M, flows from an enclosed space A with a velocity V through an orifice of area S, there is a force exerted on the surroundings in the same direction as V, which for conditions of steady flow is given by the basic laws of mechanics as (momentum force + pressure force), i.e.

$$\boldsymbol{F} = +\boldsymbol{MV} + (\boldsymbol{P}_1 - \boldsymbol{P}_0)\,\boldsymbol{S} \tag{16}$$

where P_1 is the static pressure in the jet at the exit section and P_0 the external pressure.

Thus there is an equal and opposite force of reaction applied to the enclosed space A,

$$-\boldsymbol{F} = -\boldsymbol{MV} - (\boldsymbol{P}_1 - \boldsymbol{P}_0)\,\boldsymbol{S}$$

(This equation gives the thrust on a rocket, for example.)

When a fluid in a closed system is allowed to accelerate in some *localized* region, these laws explain the origin of the forces of attraction or repulsion between certain *components* in the system. In particular, in a *deformable* system, which may include e.g. pistons, valves and spools, it often happens that if these forces are neglected, the equation of equilibrium is completely inaccurate. This discussion will be restricted to the evaluation of the momentum term, MV, since the determination of the pressure term, $(P_1 - P_0)S$, presents no major difficulties.

4.3.2. REACTION FORCE DUE TO MOMENTUM

If a homogeneous jet with parallel streamlines is formed and emitted from component A and comes to rest in component B, the force due to the change of momentum exerted on component B is given exactly by $F_1 = MV$ (the *reaction* force on A is $-MV = -F_1$). In practice, however, jets are seldom homogeneous with parallel streamlines and they rarely come completely to rest in any one component. Force F_1 is therefore the upper limit of the real force, F_R. We shall thus first determine F_1 and then evaluate F_R from F_1 in particular cases.

4.3.2.1. *Evaluation of F_1*

It was shown in Chapter 1 that the velocity V of a jet through an orifice is related to the pressure difference, ΔP, across it by the equation

$$\Delta P = \varrho \frac{V^2}{2} \tag{17}$$

(assuming a completely efficient conversion of pressure energy to kinetic energy).

It was also shown that the jet of fluid contracts as it leaves the orifice. Since it is convenient to express the relationship for the pressure drop in terms of the mean velocity, V_m (which will be less than V), and the actual area of the orifice itself, then

$$\Delta P = \xi \frac{\varrho}{2} V_m^2 = \xi \frac{\varrho}{2} \frac{Q^2}{S^2} \tag{18}$$

Multiplying eqn. (17) and (18) together term by term, and taking the square root of each term of the product gives

$$\Delta P = \sqrt{\xi}\,\frac{\varrho}{2}\,V\,\frac{Q}{S}$$

Thus, putting $\rho Q V = MV$, we have*

$$\boxed{MV = F_1 = \frac{2}{\sqrt{\xi}}\,S\,\Delta P} \tag{19}$$

Thus, for a sharp-edged orifice ($\xi = 1{\cdot}8$)

$$MV = F_1 = 1{\cdot}5\;S\,\Delta P$$

Sometimes the cross-sectional area, S, is unknown and it is necessary to substitute for V from eqn. (17)

$$V = \sqrt{\frac{2\,\Delta P}{\varrho}}$$

Multiplying this by $M = \rho Q$ gives

$$\boxed{MV = F_1 = Q\,\sqrt{2\,\varrho\,\Delta P} = Q\,\sqrt{2\,\frac{w}{g}\,\Delta P}} \tag{20}$$

This equation shows that the maximum hydraulic force, F_1, is proportional to the flow (or, more precisely, to the geometric mean of the volume flow and the mass flow, since $Q\sqrt{\rho} = \sqrt{Q\,.\,\rho Q}$), and to *the square root of the pressure drop.*

4.3.2.2. *Evaluation of* F_R

(*a*) *General case*—If there is relative movement between A and B on the axis Ox, then an important force is the component of F_1 along Ox. If θ is the angle between the direction of V and the axis Ox, this component of the force is

$$F_{1_x} = MV\cos\theta$$

If the flow is not homogeneous, we must integrate this expression for each elemental stream tube, giving

$$F_{R_x} = \int_{(M)} V\cos\theta\;\mathrm{d}M$$

But the liquid passes right through hydraulic components so that, in a control valve, for example, it passes from the body A to the spool B and then back from B to A. If this back flow from B to A is made at a speed V' which is not negligible

* Note that, for the valve shown in *Figure 4.8*, the force exerted on the cap by the liquid changes from $P(S_1 + \lambda S_2)$ when the valve is closed (static force) to $P(2/\sqrt{\xi})S_1$ when the valve is fully open (thrust of the jet). For a perfect valve ($S_2 = 0$; $\xi = 1$), when fully open, the force of the jet, $2PS_1$, is twice the pressure force when the valve is closed, PS_1.

compared with V, i.e. if the force of the jet from A to B is not completely absorbed by B, and if the angle of each elemental jet with the axis of the spool B is θ', then

$$F_{R_x} = \int_{(M)} (V \cos \theta - V' \cos \theta')\, dM$$

If the relative movement of the components A and B is a rotation about the axis Ox, the evaluation of F_{Rx} is obviously replaced by that of the torque about Ox.

(*b*) *Axisymmetrical valves*—Most components fall into this category; a typical example is shown in *Figure 4.15*. Because of the symmetry, the resultant of the elemental forces lies along the axis $F_{Rx} = F_R$. On the other hand, V, V', θ and θ' are practically uniform in the flow, so that

$$F_R = M(V \cos \theta - V' \cos \theta')$$

Thus

$$F_R = \lambda M V \cos \theta = \lambda F_1 \cos \theta \tag{21}$$

where

$$\lambda = 1 - \frac{V' \cos \theta'}{V \cos \theta}$$

F_1 can be found from eqn. (19) or (20), while the coefficient λ depends on the geometrical shape of the valve.

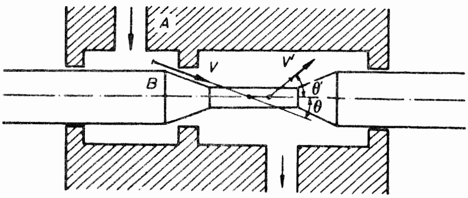


Figure 4.15

(*c*) *Application to rectangular* (*i.e. square-edged*) *valves* (cf. Chapter 10, Example 4)—The simplest valves to manufacture are those in which all the angles on the spool and the body are right angles (*Figure 4.16*). Shin Ying Lee and Blackburn[1] have shown that, when the opening e is much greater than the clearance j, and small compared with the chambers C and D

$$\theta = 69^\circ \qquad \lambda = 1$$

($\lambda = 1$ means that $V' \cos \theta'$ is negligible compared with $V \cos \theta$, which is due mainly to the low value of V' relative to V.)

Multiplying eqn. (19) and (20) by $\cos 69^\circ = 0{\cdot}36$ gives the following expressions for the hydraulic force

$$F_R = \frac{0{\cdot}72}{\sqrt{\xi}} S \, \Delta P \tag{19'}$$

and

$$F_R = 0{\cdot}52\, Q \sqrt{\frac{w}{g} \Delta P} \tag{20'}$$

It can be seen from *Figure 4.17*, however, that θ decreases as $e \to 0$; in particular, due to symmetry, $\theta = 45°$ for $e = j$, and decreases to $21°$ (i.e. $90° - 69°$) for $e \ll j$.

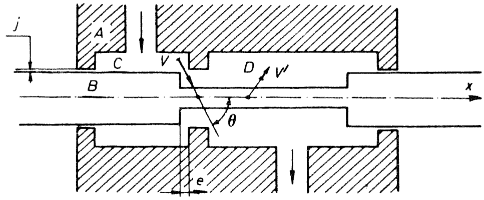


Figure 4.16

The curve of $F_R/\Delta P = f(e)$, derived from*

$$\frac{F_R}{\Delta P} = \frac{2}{\sqrt{\xi}} S \cos\theta$$

$$S = \pi De \sqrt{1 + \frac{j^2}{e^2}}$$

$$\cos\theta = g(e)$$

is shown in *Figure 4.18*, where the positive x direction corresponds to that of *Figure 4.17*.

In practice, when e is very small, the flow generally becomes laminar, so that the real values of $F_R/\Delta P$ are very much lower than those shown on the curve. Although the slope of $F_R/\Delta P = f(e)$ is generally positive, it can nevertheless become negative in the neighbourhood of $e = j$. The conclusions to be drawn from this will be discussed below; it will be shown that this irregularity of the curve is the main disadvantage of the use of these valves with rectangular profiles, which are so easy to manufacture.

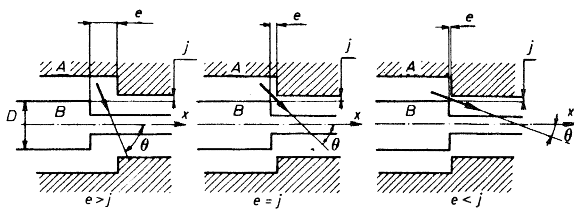


Figure 4.17

Note 1—With the sign convention of *Figure 4.17*, the direction of F_R is positive along Ox. In the sense of the opening of the valve, however (e increases as the spool is moved in the $-x$ direction), the direction of F_R is negative. *Therefore, the force F_R tends to close the valve.* In *Figure 4.18* it would be preferable to consider $F_R/\Delta P$ as positive in the direction of increasing e, which would mean drawing the curve on the other side of Oe, thus inverting the ordinate and slope

* See the value of ξ in Section 1.4.1.2.

of the curve for each value of e. The customary way of presenting this graph, however, is as given (see the discussion on this subject in Section 4.5).

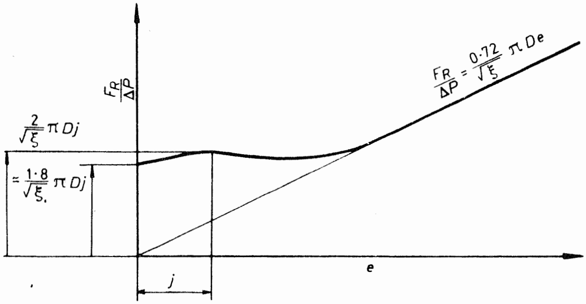


Figure 4.18

Note 2—Reversal of flow direction: (*a*) *Rectangular valve slightly open*—Consider a rectangular valve with an opening small compared with the size of the chambers C and D, as shown in *Figure 4.19*. For a small element of fluid subtending an angle $d\psi$ to the axis, the angles of convergence and divergence are symmetrical with respect to O, the mid-point of the line joining a and b. The result of changing the direction of flow is therefore to change the angle θ to $\theta^* = \pi - \theta$.

If, for the first direction of flow (direction 1), the force exerted on the downstream element B (action) is given by

$$F_R = + F_1 \cos \theta$$

then, for the opposite direction (direction 2), the force exerted on the downstream component, which is now A, is $+F_1 \cos(\pi - \theta)$ and the force exerted on the upstream component, which is now B (reaction), is given by

$$- F_1 \cos(\pi - \theta) = + F_1 \cos \theta = + F_R$$

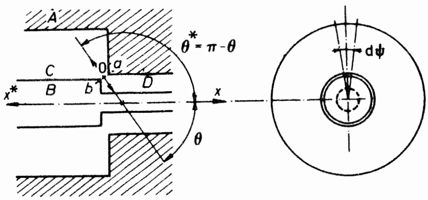


Figure 4.19

Thus the change in flow direction through a slightly open rectangular valve does not affect the magnitude or the sign of the hydraulic force on the valve components. (This force always tends to close the valve.)

(*b*) *General case*—In the case of a non-rectangular valve, or a rectangular valve which is fully open, the convergent–divergent region is no longer symmetrical about point O. The flow angle depends mainly on the upstream (convergent) rather than on the downstream (divergent) shape, and so θ^* is usually not equal to $\pi - \theta$. *A change in flow direction alters* F_R.

4.3.3. VALVE OSCILLATIONS

It has been shown that, for a rectangular valve, the flow exerts a closing force, F_R, between the components A and B and that $F_R/\Delta P$ increases with e, except near $e = j$. This is true for the majority of non-rectangular valves, regardless of the profile used.

ΔP, however, often decreases with the rate of flow, and therefore with e, owing to the fact that, in the upstream characteristics, P decreases with Q, and in the downstream characteristics, P increases with Q. The result is that the curve

$$F_R = g(e) = \frac{F_R}{\Delta P} \cdot \Delta P$$

has a region with a negative slope which is more pronounced than that of the $F_R/\Delta P = f(e)$. The magnitude of the slope will depend on the upstream and curve downstream impedances.

Now the condition for static stability of a point which can move on a straight line, the x axis for example, and which is subjected to a force ϕ, a function of x, is

$$\frac{\mathrm{d}\Phi}{\mathrm{d}x} < 0 \tag{22}$$

where a positive force acts in the positive x direction.

Therefore, if we take the positive force as acting in the negative x direction, as we did above, so that a restoring force* is positive (restoring force, $F = -\phi$)

$$\frac{\mathrm{d}F}{\mathrm{d}x} > 0$$

If the component B is subjected only to a constant external force, K, in addition to F_R, it will therefore be in stable equilibrium at all points other than for values of e close to j.

The oscillations which take place in many valves when set in motion near the closed position are evidence that such regions of instability exist. These vibrations may damage the valve. The instability can be eliminated by the application of an external restoring force, F_e, to the movable element of the valve (by a spring, for example), so that $\mathrm{d}F_e/\mathrm{d}e + \mathrm{d}F_R/\mathrm{d}e$ is always positive.

In order to evaluate exactly the required force F_e, it is necessary to know the relationship $F_R = f(e)$. This evaluation is very complicated. Note that stable equilibrium is attained if

$$\frac{\mathrm{d}F_e}{\mathrm{d}e} \geqslant \frac{\mathrm{d}F_1}{\mathrm{d}e}$$

where F_1 is the maximum theoretical force defined in eqn. (19) and (20)

$$F_1 = \frac{2}{\sqrt{\xi}} S\Delta P = Q\sqrt{2\frac{w}{g}\Delta P})$$

* On this subject, see Section 4.5 and Chapter 10, Example 5.

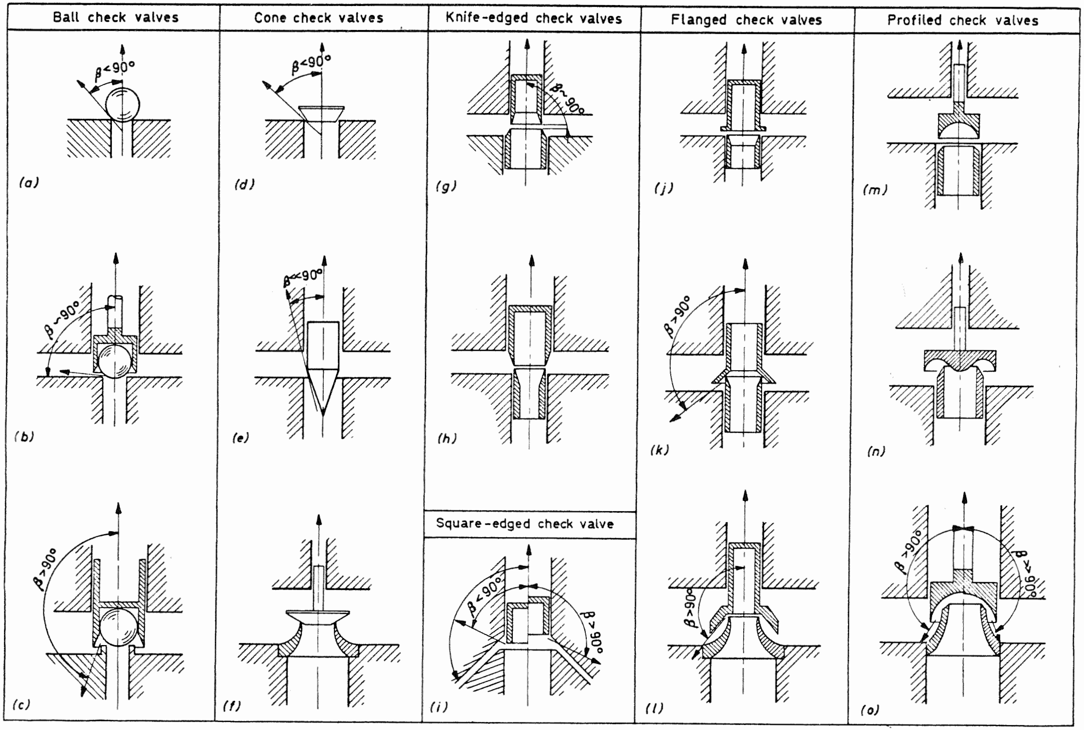


Table 4.1. Check valves[2]

(a) Normal valve, $\beta < 90°$; (b) valve with ball and deflector, $\beta \sim 90°$; (c) valve with ball, deflector and raised seating, $\beta > 90°$; (d) normal valve, $\beta < 90°$; (e) conical angle very small, β very small (conserves hydraulic closing force, see Section 4.3.4.3b); (f) valve with streamlined nozzle, $\beta < 90°$; (g) normal arrangement: large openings, $\beta \sim 90°$, small openings, $\beta < 90°$ (cf. cross-reference under e); (h) inverted arrangement: large openings, $\beta \sim 90°$, small openings, $\beta > 90°$; (i) form of outlet causes β to increase as valve is opened; (j) normal arrangement, similar to g (better at small openings); (k) valve with single flange, $\beta > 90°$ (note static equilibrium); (l) valve with flange deflector and nozzle, $\beta > 90°$; (m) valve with guide and recessed piston; (n) small openings, $\beta < 90°$, large openings, $\beta > 90°$; (o) $\beta > 90°$, may reach or even exceed 150°.

This result is only given as a guide. It is obvious that the minimum value of $dF\theta/de$ necessary depends on the shape of the valve and the impedances of the upstream and downstream circuits. Many valves are stable with lower values of the restoring stiffness (or even zero stiffness).

Table 4.1 gives a number of examples of available or proposed designs of check valves and indicates the direction of the fluid jet at the outlet of each valve. (In the diagrams, the movable part is always the top part and the angle β is that between the fluid jet and the positive direction of the valve opening.)

Note—Clearly the brief analysis given does not completely resolve the problem of valve vibrations. The problem is, in fact, extremely complex and depends upon a large number of parameters. This is illustrated by frequently successful methods of eliminating valve vibrations, such as the practice of artificially increasing the pressure in the outlet chamber of the valve by even a few lb./in.^{2}

4.3.4. THE PROBLEM OF BALANCING

Given a valve in which the hydraulic forces are excessively high, we can
(1) increase the control force, F_c, so as to decrease the ratio F_R/F_c
(2) compensate F_R by an artificial external force
(3) decrease or nullify the hydraulic force F_R itself.

In order to do this, note that eqn. (21) gives $F_R = \lambda F_1 \cos\theta$, so that we can either increase θ, if possible to $\pi/2$, or decrease λ, if possible to zero.

4.3.4.1. *Increasing the control force*

A simple solution is to increase the physical dimensions of the control parts of the valve (for example, the pistons and springs of the unloading or throttling valves discussed in Section 3.5.3 and shown in *Figures 3.38* and *3.40*). This solution obviously has a deterimental effect on the weight, size and response speed of the equipment.

Another solution is to introduce a device which magnifies the force between the control element and the valve spool. *Figure 4.20* is a diagrammatic sketch of a 'differential piston follower' which is often used for this purpose. This

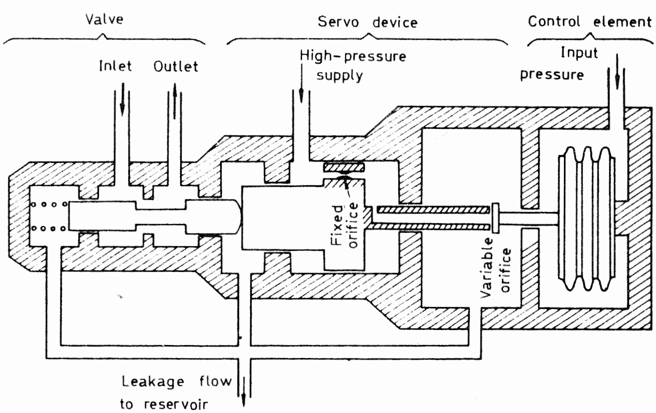


Figure 4.20

solution is fairly complex and expensive but usually reliable. (It is also expensive with regard to energy—the quantity of liquid necessary for the servo flow of such a hydraulic control cannot always be spared.) This piston is useful, however, for large flows at high pressure. In practice, it is indispensable when the energy level in the input is very low, as in the case of aircraft servo valves which control the flow of hydraulic fluid to the actuators of the ailerons or elevators, with power of the order of 0·05 to 0·1 W ($\hat{=}$10 mA at about 10 V) available as the output from the automatic pilot.

4.3.4.2. *Compensation of the hydraulic force, F_R*

Compensation of the hydraulic force, i.e. introducing an external force which is equal and opposite to it, is a very difficult procedure, especially if F_R does not vary linearly with e (as in the rectangular valve of *Figure 4.18*), or the valve operates under a varying pressure difference, ΔP, in particular if ΔP is not a simple, single-valued function of the aperture or the rate of flow. For these reasons this solution is seldom used.

4.3.4.3. *Decreasing or nullifying the hydraulic force, F_R, itself*

Apparently no method of calculation exists which is both simple and accurate enough to be applied industrially. However, a few simple theoretical considerations enable us to reduce appreciably the number of tests and modifications required for the perfection of a valve.

Two solutions will be examined, corresponding (cf. eqn. 21) to increasing θ (V-shaped spool valves) or, alternatively, decreasing λ (cone valves). (The rectangular valve, despite its ease of manufacture, is usually rejected as being too difficult to balance when this problem is tackled seriously.)

(*a*) *V-shaped spool valve*—A sketch of this type of valve is given in *Figure 4.21*. The chamfering of the edges decreases the value of cos θ.

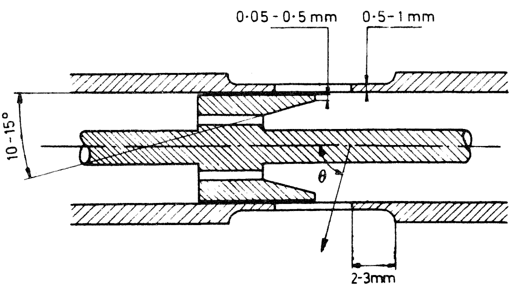


Figure 4.21

The disadvantages of this type of valve are that the edges are easily damaged, and the fact that, at small openings, their operation is similar to that of the rectangular valve. This disadvantage can be reduced by using a particular design where the knife edge takes the form of a spiral and the holes in the cylinder are staggered, but unfortunately this reduces the gain, $\Delta Q/\Delta e$. This type of valve is therefore generally used only in equipment where it does not have to function near the closed position.

(*b*) *Cone valve*—In practice, F_R can be completely balanced only when the flow is not subjected to the abrupt variations in flow angle which are present in the rectangular and V-shaped spool valves when operating near the closed position.

It is obvious that, for a conical spool with a fairly low total angle 2α (*Figure 4.22*), the final flow angle, θ, is always near zero. Practical results confirm this for $\alpha = 15°$.

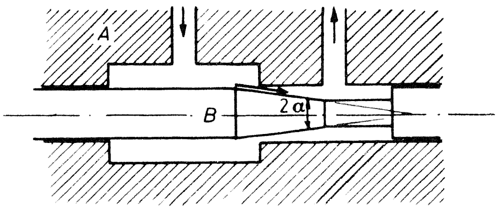


Figure 4.22

We know that the hydraulic force, F_R, is exerted on component B because the stream of fluid set in motion in component A loses its kinetic energy on component B. We will therefore try to direct this stream of fluid in such a way that it will lose its kinetic energy *on component* A, i.e. we must nullify the coefficient λ in eqn. (21) by creating a term $V' \cos \theta'$ equal to the term $V \cos \theta$.

It can be shown that F_R can be reduced to zero by using certain profiles of the spool and chambers of the type shown in *Figure 4.23a*. In addition, the use of a symmetrical profile (*Figure 4.23b*) overbalances the valve, so that *the sign of* F_R *is changed* (and decreased slightly in magnitude). In practice, the design of convergent–divergent profile of the type shown in *Figure 4.23a* is difficult, due chiefly to the lack of a satisfactory method of analysis of the flow involved.

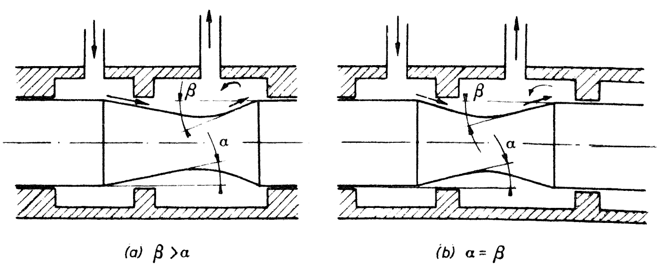


Figure 4.23

There is, however, an alternative design which gives excellent results. It consists of a double valve (*Figure 4.24*) having an overbalanced portion with a symmetrical profile, I, together with an unbalanced portion with a conical profile, II. The flow direction is different for these two profiles, and the adjustment of the valve characteristics can be made by altering the conical profile II only.

If the suffixes I and II refer to the flows over profiles I and II, respectively, the hydraulic force is given by

$$F_R = \lambda_I M_I V_I \cos \theta_I + \lambda_{II} M_{II} V_{II} \cos \theta_{II}$$

Now

(*a*) $V_I = V_{II} = V = \sqrt{\dfrac{2\,\Delta P}{\varrho}}$

(*b*) the mass flows are a function* of the angles α and γ:

$$M_I = M\left(\frac{\alpha}{\alpha + \gamma}\right) \qquad M_{II} = M\left(\frac{\gamma}{\alpha + \gamma}\right)$$

(*c*) $\cos \theta_I$ and $\cos \theta_{II}$ are approximately 1 (since α and γ are small)
(*d*) λ_{II} is approximately 1.

F_R therefore reduces to

$$F_R = M\,V\left(\lambda_I \frac{\alpha}{\alpha + \gamma} + \frac{\gamma}{\alpha + \gamma}\right)$$

so that $F_R = 0$ if $\quad \gamma = -\lambda_I \alpha$

(λ_I is generally of the order of $-0{\cdot}6$ to $-0{\cdot}8$).

In practice, the design procedure is as follows:

(1) make a spool with γ too small, for example $\gamma = \gamma_0 = 0{\cdot}5\alpha$
(2) measure F_R and evaluate

$$\lambda_I = \frac{F_R}{M\,V}\,\frac{\alpha + \gamma_0}{\alpha} - \frac{\gamma_0}{\alpha}$$

(3) alter the angle γ so that $\gamma = -\lambda_I \alpha$
(4) repeat until γ is correctly adjusted.

A single test and readjustment is usually sufficient, but it is necessary since the value of λ varies from valve to valve.

The main difficulty lies in making the lengths *a* and *b* (*Figure 4.24*) absolutely accurate.

Figure 4.25 outlines possible faults that may occur.

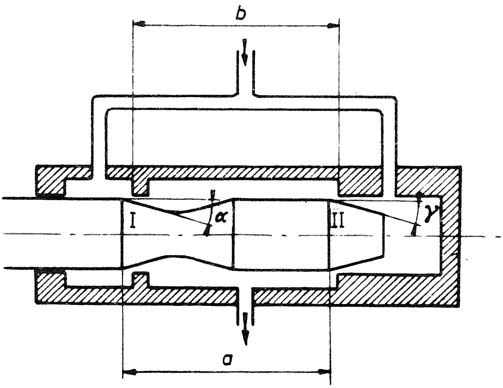


Figure 4.24

* For conical spools with small total conical angles, the area of flow is proportional to these expressions.

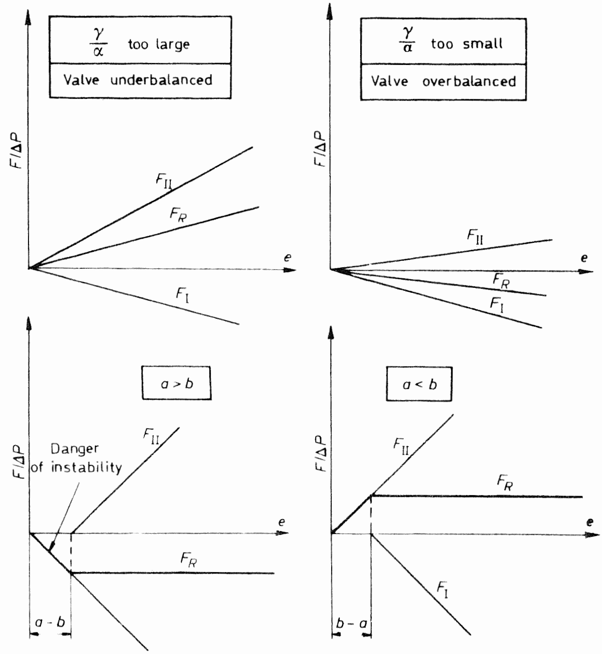


Figure 4.25

4.3.5. DESCRIPTION OF AN EXPERIMENTAL METHOD OF BALANCING

In practice, it is difficult to measure the forces exerted on a spool submerged in a liquid, or even its exact position. Some American universities and companies have designed very complex and expensive apparatus for these experiments. But it is often possible to deduce all the necessary information from a few simple hydraulic measurements.

Take as an example the design of an unloading (or by-pass) valve as described in Section 3.5.3 and shown diagrammatically in *Figure 3.38*.

The curves given below refer to a valve of the 'double-cone' type, but the method given is independent of the type of valve. The experimental layout is shown in *Figure 4.26*. An adjustable orifice, O_1, which represents the 'control section' through which the valve should ensure a constant loss of head, is placed in parallel with the unloading valve across a positive displacement pump. The second adjustable orifice, O_2, represents the downstream impedance. The loss of head, $P_1 - P_2$, of orifice O_1 is applied across the diaphragm of the unloading valve.

We measure: the pressures P_1 and P_2 upstream and downstream of O_1; the flow through the valve, Q_b (the by-pass flow); and finally, the pressure P_0 at the valve outlet, if it is not zero.

The valve opening can be expressed as a function of the reduced flow knowing the geometric shape $[S = f(e)]$ and the coefficient of loss of head for the valve

$$\left[P_1 - P_0 = \xi \frac{\varrho}{2} \frac{Q^2}{S^2}\right]$$

The force, $F_T = (P_1 - P_2)S_e$, can be evaluated knowing the effective surface area, S_e, of the diaphragm.

The force $F_R = T + Re$ developed by the spring can be calculated knowing the initial force, T, and the stiffness, R.

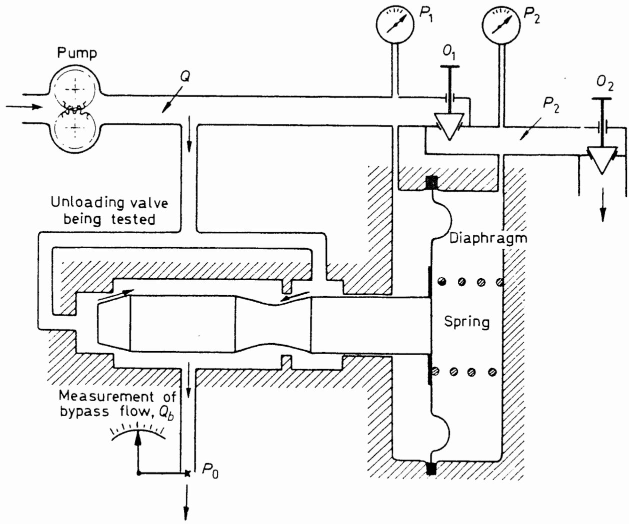


Figure 4.26

Finally, the required hydraulic force is given by

$$F_R = F_T - F_r$$

But there is an alternative approach. By varying both Q_b and $(P_1 - P_0)$ by adjusting the two orifices O_1 and O_2 and also by changing the speed of the pump, we can plot curves of equal Q_b and equal $(P_1 - P_0)$ on the axes:

$$\begin{cases} x = \dfrac{Q_b}{\sqrt{P_1 - P_0}} \\ y = P_1 - P_2 \end{cases}$$

For a balanced valve, these curves are coincident.

In effect, a value of $Q_b/\sqrt{P_1 - P_0}$ corresponds to a single value of e, and therefore to a single value of the force of the spring $F_r = T + Re$ and a single value of $(P_1 - P_2)$ since, if $F_R = 0$, $P_1 - P_2 = F_T/S_e = F_R/S_e$.

For a valve which is not balanced, suppose (*Figure 4.27*) that the ordinate

$$x_i = \left(\frac{Q_b}{\sqrt{P_1 - P_0}}\right)_i$$

intersects the curves

$$P_1 - P_0 = (P_1 - P_0)_i$$

and

$$P_1 - P_0 = (P_1 - P_0)_j = (P_1 - P_0)_i + \Delta(P_1 - P_0)$$

at values of the y ordinate given by

$$P_1 - P_2 = (P_1 - P_2)_i$$

and
$$P_1 - P_2 = (P_1 - P_2)_j = (P_1 - P_2)_j + \Delta(P_1 - P_2)$$

The hydraulic force, F_R, is increased by $\Delta F_R = \Delta(P_1 - P_2)S_e$ between these two points, since F_r is constant.

If λ and θ do not change, using eqn. (19) and (21), we have

$$F_R = K \,.\, S\,(P_1 - P_0)$$

Thus, with S constant ($Q_b/\sqrt{P_1 - P_0}$ constant)

$$\frac{\Delta F_R}{F_R} = \frac{\Delta(P_1 - P_0)}{P_1 - P_0}$$

and finally

$$F_R = \frac{\Delta(P_1 - P_2)}{\Delta(P_1 - P_0)}\, S_e\,(P_1 - P_0)$$

For a valve which is not balanced, the shape of the curves $P_1 - P_0$ = constant shows whether there is any danger of instability, since the slope of the curves $P_1 - P_2 = fQ/\sqrt{P_1 - P_0}$ at $P_1 - P_0$ = constant has the same sign as the slope of $F_R/(P_1 - P_0) = f(e)$.

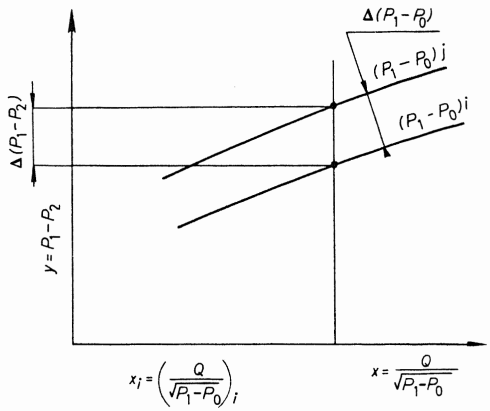


Figure 4.27

It should be noted that, as stated at the beginning of Section 4.3.3, the stability of the system depends on the characteristics of the pump and the circuit downstream. This is why the region of instability *observed* on the experimental curves given in *Appendix 4.1* is slightly more extensive than that of negative slope on the curves of constant $(P_1 - P_0)$.

This fact is very important; it often explains the difference of behaviour in a system under test conditions and in its final surroundings, or between the same system under two different test conditions.

4.4. FORCES DEVELOPED BY SEAL ARRANGEMENTS

4.4.1. TYPES OF SEAL AND FORCES DEVELOPED

Sealing problems are always present in hydraulic systems. There are three different types: the prevention of leaks from the system to its environment; the separation of two different types of fluid (petrol and oil, oil and air, etc.); and the separation of two chambers filled with the same fluid but at different pressures.

Various methods can be used to provide a seal between two components. If these are in relative motion, the presence of a seal usually produces *forces* which disturb the conditions of equilibrium, their nature and magnitude depending on the type of seal used.

All seals can be reduced to three basic types. These types, together with the nature and magnitude of the associated forces, are set out in *Table 4.2*.

Table 4.2. Types of seals

Type of seal	*Frictional force*	*Elastic restoring force*	*Disadvantages*
(1) Reduction of the clearance between two undeformable surfaces	very small if surfaces well finished	zero	seal is not perfect*
(2) Joining the two components by a deformable fluid-tight partition	zero (except for hysteresis of partition)	depends on stiffness of deformable partition	unsuitable for relative motions of high amplitude, in particular with rotation of no angular limit; very cumbersome
(3) Placing a plastic seal† between the components, attached to one surface and rubbing against the other	large: 10 or 100 times greater than without the seal‡	restoring force can exist near position of equilibrium in a small oscillation corresponding to elastic deformation of seal	large friction force; seal less efficient than the preceding

* It is often possible to improve the seal by certain devices: staged decompression chambers, artificial counter-pressure, introduction of air between the parts to avoid mixing.

† This type can be considered as intermediate between (1) and (2). A *flexible* seal is efficient but has high friction forces; a *metal* seal is not so efficient but has a lower friction than the plastic one.

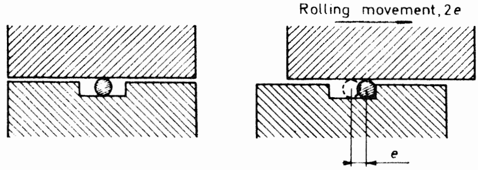
‡ The friction force depends on:
the *material* of which the seal is made (Teflon, for example, can reduce it by a factor of 4, 6 or even 8, provided certain precautions are taken to allow for the low elasticity and the tendency to extrusion);
the *form* and *tightness* of the seal (a lipped seal has less friction than a ring seal of the same initial degree of tightness, but this difference disappears as the pressure difference increases);
the *nature and state of the surface* of the seal;
the *nature of the fluid* which wets the seal;
the *pressure of the fluid*.
Note that for ring seals operating near the equilibrium position, where ΔP is very small, there is a region of discontinuity within which the friction force is almost zero, due to the seal rolling without slipping (*Figure 4.28*).

4.4.2. EFFECTS ON THE DESIGN OF HYDRAULIC EQUIPMENT

Table 4.2 shows that the choice of the type of seal depends on the allowable magnitude of the frictional and elastic forces; the degree of fluid-tightness required; the nature of the relative motion between the two parts, and the space available.

We shall not consider the seals in *power components* (rams, motors), since the allowable magnitude of the forces is sufficiently high to permit any type of seal to be used. We shall consider, however, the seals in *sensing components* (detectors, adders, control valves, etc.).

Figure 4.28



In these components it is *essential that frictional forces be avoided at all cost.* It is absolutely impossible to compensate for them; they create a discontinuity which remains in the whole sensing chain.

On the other hand, *elastic restoring forces* are much less of a nuisance, since they can be compensated (by increasing the gain downstream, integration, etc.).

Often devices which, at first sight, seem to be unnecessarily complicated, are used for the purpose of avoiding frictional forces.

We will briefly examine the more usual methods used.

1. *Evading the problem*

A layout is adopted which does not require a seal, or which places it elsewhere, i.e. between two fixed parts or in a power component.

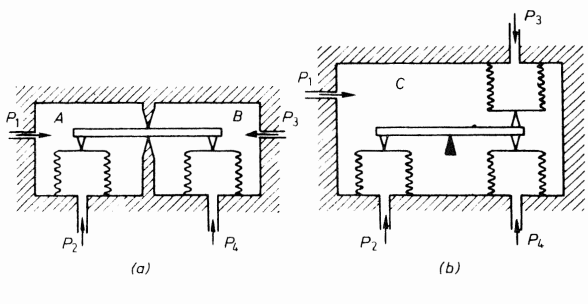


Figure 4.29

First example—Tachometric regulators immersed in the fuel: The whole regulator is immersed in fuel at high or low pressure. The seal with the exterior is made on the shaft measuring the speed of rotation upstream of the regulator.

Second example—Servo valves with wet motors: The windings of the electric motor are immersed in low-pressure oil. The external seal is made where the connecting wires enter the casing.

Third example—Measurement of the algebraic sum of four pressures, i.e. to find $(P_1-P_2)-(P_3-P_4)$. One solution is to place a capsule under pressure, P_2, in a chamber A at a pressure P_1, and a second capsule under pressure P_4 in another chamber, B, at a pressure P_3 (*Figure 4.29a*). It is difficult to achieve a satisfactory seal around the lever arm between A and B. This problem is eliminated, if three capsules under pressures P_2, P_3 and P_4 are used in a single chamber at a pressure P_1, as shown in *Figure 4.29b*.

2. *Reduction of seal effectiveness*

A small leakage value in the calculation is often permissible, e.g. from high to low pressure in a control valve. Sometimes, when it is important to avoid the mixing of two fluids, a communal container is placed between them to collect the leakage.

In these devices, it is often preferable to operate in rotation rather than translation, since (*a*) for equal clearance (i.e. equal leakage) the friction is less; (*b*) the parasitic axial force, $s(P_A - P_B)$, can be absorbed by a thrust bearing.

These advantages of rotation over translation are illustrated in *Figures 4.30a* and *4.30b* which show the equivalent rotational and translational solutions to the same problem, the introduction of a control parameter in the fuel regulator, representing the difference $P_1 - P_0$ between the two *air pressures*.

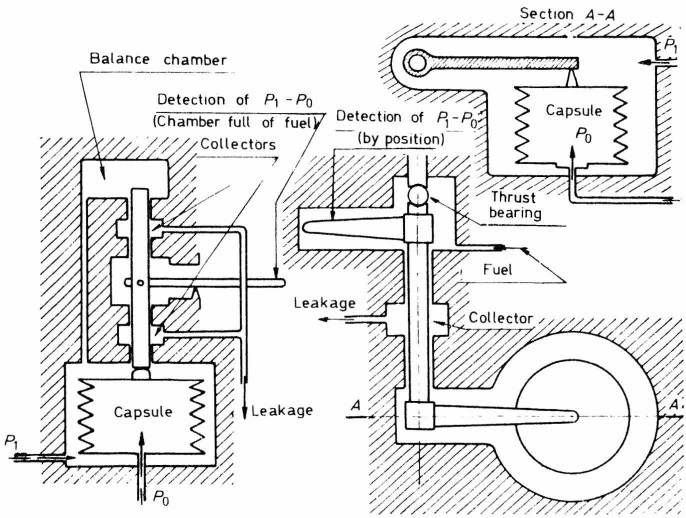


Figure 4.30. (*a*) Translational arrangement; (*b*) rotational arrangement

(*a*) Note the presence of the balance chamber and the two leakage collectors; (*b*) the balance chamber is replaced by a thrust bearing and there is a single leakage collector.

3. *The solution using a deformable wall*

This method does not permit large relative movements. For this reason, the detection of force is normally used instead of position (see Section 4.2.3.2).

The difficulty of evaluating the effective surface area of a diaphragm exactly renders the problem of balancing the pressure forces difficult and often results in a replacement of translational movements by rotation, in this case as well as the previous one.

Figure 4.31 shows this method applied to the same problem of introducing the parameter $P_1 - P_0$ to a fuel regulator.

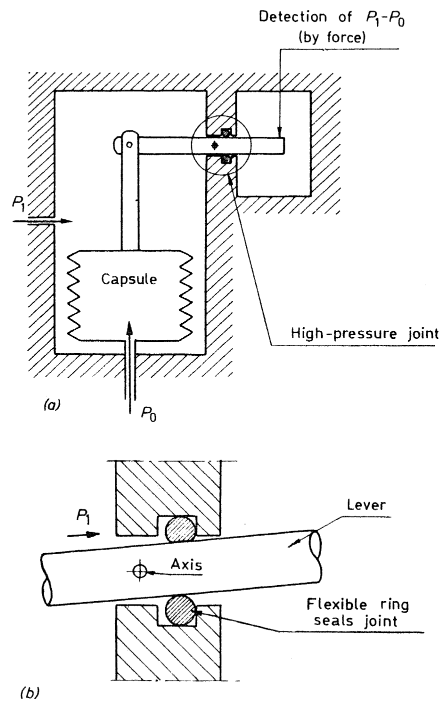


Figure 4.31. Force detection of pressure difference (Lucas); (*a*) rotational movement; (*b*) detail of high-pressure joint

4.5. FORCES AND STIFFNESSES

The inaccuracy of the nomenclature of this subject obscures even simple concepts.

4.5.1. DEFINITIONS AND SIGN CONVENTIONS

A force F, whose magnitude is a linear function of the x ordinate of its point of application, is defined by the vectorial relationship

$$\boldsymbol{F} = \boldsymbol{F}_0 + R\,.\,\boldsymbol{Ox} \tag{23}$$

where $\boldsymbol{F}_0$ is the vector corresponding to the initial force (dimension, F) and R is a *scalar quantity* called the stiffness (dimension, FL^{-1}).

The equilibrium position, $x_0 = -\boldsymbol{F}_0/R$, of a body subjected to the force F will be stable if there is a force tending to return it to its original position when it is deflected slightly from that position, i.e. if $F = R(x-x_0)$ has the opposite sign to that of $(x-x_0)$.

The condition for *static stability* is therefore

$$\boxed{R < 0}$$

(or $\Sigma R < 0$ if the point is subjected to several forces).

The stiffness forces encountered in practice are usually negative, so that, in order to use positive numbers, it is customary to *consider forces as positive in the sense of negative x.*

Thus, with $f_0 = -F_0$ and $r = -R$

$$f = -F = f_0 + rx$$

The condition for static stability therefore becomes $r > 0$.

The convenience of using this nomenclature is permissible only if we define f and r as the *restoring* force and stiffness, respectively.

4.5.2. THE SIGN OF STIFFNESSES

The stiffness, R, is a scalar quantity and is independent of the choice of positive axis. Obviously this also applies to the restoring stiffness, $r = -R$. Thus, if a force F is produced by a component A, the stiffness is a characteristic coefficient of the component A, and it can be defined in magnitude and sign.

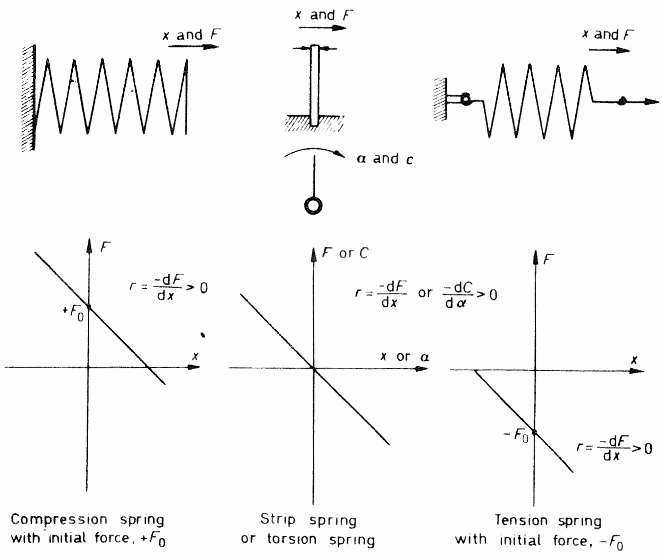


Figure 4.32

Stiffness of mechanical elements—By definition, every stable mechanical element should have a positive restoring stiffness, r. This is the case for coiled springs in tension or compression, strip springs and torsion springs, as shown in *Figure 4.32*.

Consequences

(1) The addition of a normal spring ($r > 0$) to a moving system will always increase the static stability. The initial force in the spring only affects the actual position of equilibrium.

(2) If it is required to make a system which is subjected to n forces more sensitive by decreasing the restoring stiffness, it is necessary to add a supplementary element which is *unstable* or, more precisely, used in an unstable region, ($r < 0$). The practical construction of such a component is delicate.

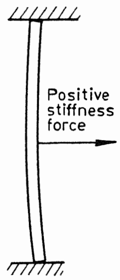


Figure 4.33

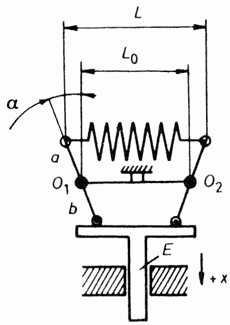


Figure 4.34

We can use a strut precompressed beyond its yield point (*Figure 4.33*) or, even better, a device of the type shown in *Figure 4.34*.

A tension spring of restoring stiffness, K, and free length, L_0, is fastened to the ends of two small movable rods pin-jointed at fixed points, O_1 and O_2 distance L_0 apart. The other ends of the rods are fitted with rollers and apply force F to the flat plate E; a and b are the lever lengths defined in *Figure 4.34*, α is the inclination of the rods to the vertical, and the origin of x is chosen so that $x = 0$ when $\alpha = \pi/2$ (position of unstable equilibrium). Taking moments about O_1, we have

$$K(L - L_0)\, a \cos \alpha = \tfrac{1}{2} Fb \sin \alpha$$

But since $L - L_0 = 2a \sin \alpha$,

$$F = \frac{4\,Ka^2}{b} \cos \alpha$$

Putting $x = b \cos \alpha$,

$$F = 4\,K \frac{a^2}{b^2}\, x$$

i.e. the restoring stiffness, $r = -4Ka^2/b^2$, is *constant* and *negative*.

Hydraulic stiffnesses—We have seen that the restoring stiffnesses of non-balanced control valves are usually positive except near the fully closed position. We found that the static hydraulic forces in these valves are compensated. This is not so for flapper-and-nozzle valves (cf. Section 3.4.2), and this point must be clarified.

The normal hydraulic potentiometer (i.e. with flapper-and-nozzle valve) has a positive restoring stiffness. It can be seen in *Figure 4.35* that the positive displacement of the flapper plate opens the orifice and causes the pressure P to drop, thus decreasing the force $P\,.\,s$.

On the very rare occasions when a potentiometer having a negative restoring stiffness (unstable potentiometer) is required, to achieve an irreversible hydraulic lock for example, it is sufficient either to reverse the flow direction or to adopt a special flapper arrangement, as shown in *Figure 4.36.*

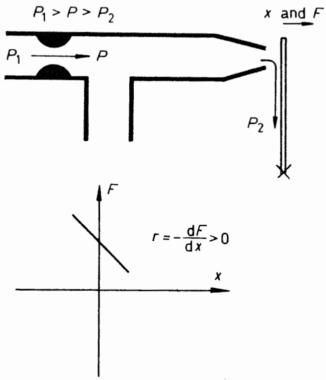


Figure 4.35

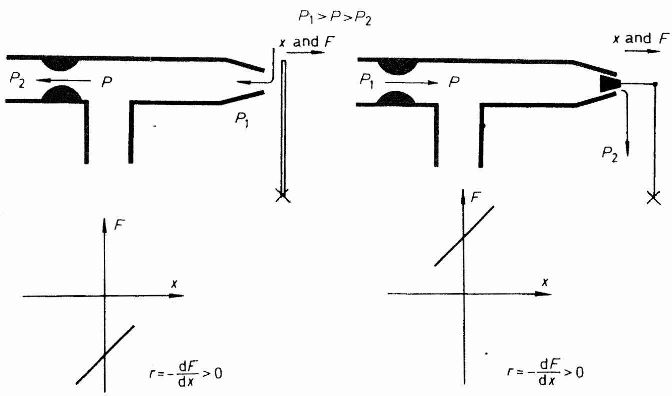


Figure 4.36

4.5.3. ORDER OF MAGNITUDE OF HYDRAULIC STIFFNESSES

The hydraulic stiffnesses of control valves with compensated static hydraulic forces have been set out in Section 4.3.

For hydraulic valves with uncompensated static hydraulic forces (i.e. flapper-and-nozzle valves), the values of the stiffness are often considerable and very much higher than the stiffness of the springs which are generally associated with them, although the values of the mechanical and hydraulic *forces* are of the same order. This arises from extremely small displacements causing large variations of pressure and, therefore, of hydraulic force.

Thus, the flapper-and-nozzle valve of Section 3.4.2, with $P_1 = 100\ \text{kg/cm}^2$

and $P_0 = 0$, has a restoring hydraulic stiffness, at $P = 50$ kg/cm², of

$$r = \frac{0{\cdot}040}{0{\cdot}0065} = 6{\cdot}1 \text{ kg/mm}$$

since a flapper plate displacement of $6{\cdot}5\ \mu$ changes the pressure P from 50 to 55 kg/cm², and hence $F = P\sigma_2$ from 0·40 to 0·44 kg. The restoring stiffness of the attached spring is of the order of 0·1 to 0·2 kg/mm.

APPENDIX 4.1

EXAMPLES OF EXPERIMENTAL INVESTIGATIONS OF DYNAMIC HYDRAULIC FORCES

1. DESIGN OF AN UNLOADING VALVE

Figures 4.37a and *4.37b* show the sectional details, drawn to scale, of a valve before and after final adjustment of the design. The modifications are summarized below.

	Before	*After*
C	No chamfer	30° chamfer
L	Strut, $l = 14$	Strut, $l' = 13$
H_1	4 holes, diam. 4	4 holes, diam. 6
H_2	None	4 holes, diam. 6

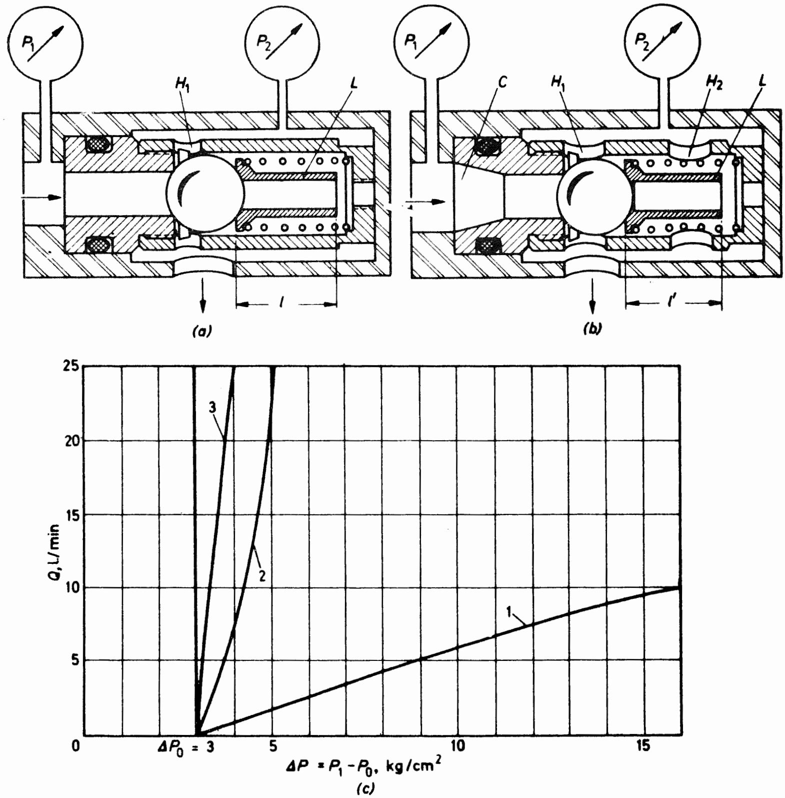


Figure 4.37

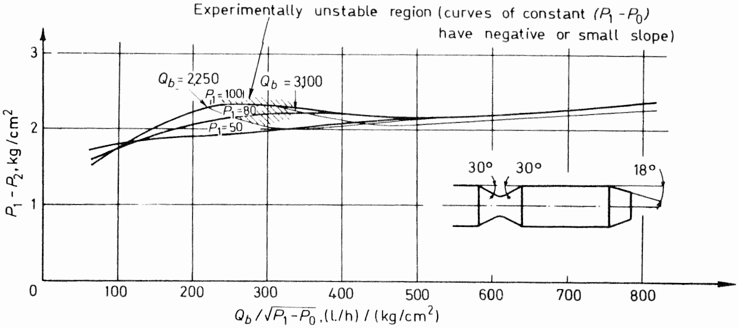
Figure 4.37c gives the flow-pressure characteristics for both designs, together with the theoretical characteristic estimated neglecting hydraulic forces. Note the considerable improvement achieved by the modification and the difference still existing between the final design and the theoretical estimate.

Thus for $P_0 = 3$ kg/cm^2, the values of ΔP (kg/cm^2) required to produce a flow of 10 l./min are

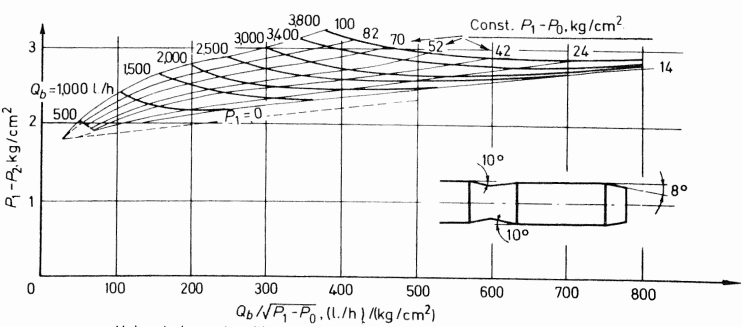
(1) before modification: $\Delta P = 16 = P_0 + 13$
(2) after modification: $\Delta P = 4{\cdot}3 = P_0 + 1{\cdot}3$
(3) according to theory: $\Delta P = 3{\cdot}4 = P_0 + 0{\cdot}4$

Tests on intermediate designs, which are not given here, indicated that the improvement in performance was mainly due to the insertion of the holes H_2. This means that in the initial design the static pressure in the closed chamber containing the spring was appreciably increased by the flow around the ball. The holes H_2 equalized the pressures on each side of the sleeve. It would have been interesting to have refined the design of the valve further in an attempt to achieve the theoretical estimate, but this was not possible owing to lack of time and production requirements.

2. EQUILIBRIUM OF A DOUBLE CONICAL UNLOADING VALVE



Balance correct, except near the closed position (angles too large)



Valve balanced, with no region of instability (small angles)

Figure 4.38 (*a*) Effect of the angles;

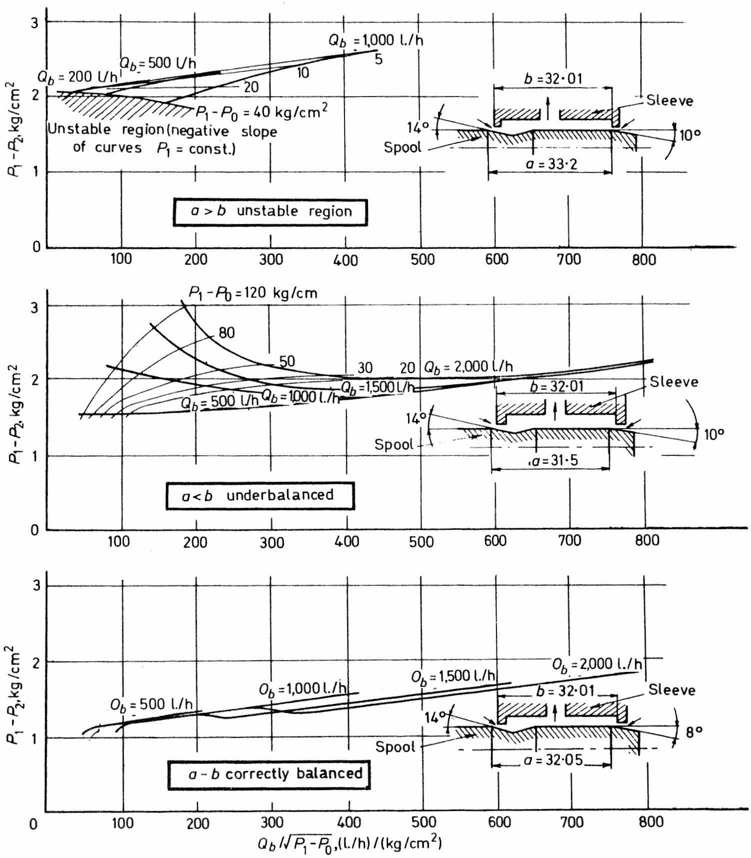


Figure 4.38(b) effect of a and b (see page 118)

APPENDIX 4.2

VALVE STICKING[3]

1. INTRODUCTION

One of the most common troubles encountered in the use of valves is that of sticking. This may occur when the valve remains at rest under pressure for a few moments. A higher force is required for initial movement than can be accounted for by normal operational resistance.

Since this effect disappears when the pressure in the valve drops, it can only be due to the presence of an asymmetrical pressure distribution around the circumference of the spool which presses it against the sleeve. The resultant axial resistance is therefore the product of this radial force and the coefficient of friction for the materials of the spool and the sleeve, while the time intervals necessary for the appearance and disappearance of sticking correspond to the times necessary for breaking the compressed oil film and for re-establishing it.

This phenomenon is considered in the analysis outlined below which gives an estimate of the magnitude of the effect and indicates the ways in which it may be reduced to a minimum.

2. PRELIMINARY ANALYSIS

Poiseuille's equation for the one-dimensional flow between two adjacent parallel planes (see Appendix 1.4) is

$$\Delta P = \frac{12\mu L Q}{e^3 l} \quad \text{or} \quad Q = \frac{\Delta P l e^3}{12\mu L} \tag{1}$$

where

L, l = length and width of the section considered
e = clearance between the planes
μ = absolute viscosity
Q = volume flow
ΔP = loss of head.

Now suppose that the planes are not parallel and that the clearance, y, is a function of x (see *Figure 4.39*), i.e. $y = ax+b$. If the loss of head between x and $x+\mathrm{d}x$ is $\mathrm{d}P$, we have

$$\mathrm{d}P = \frac{-12\mu Q\,\mathrm{d}x}{l y^3}$$

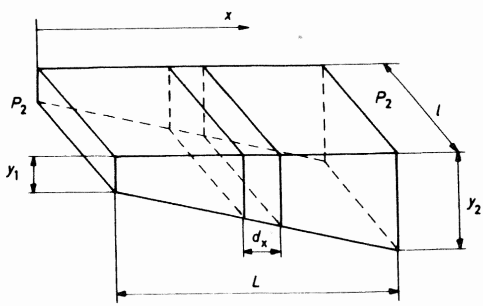


Figure 4.39

Now Q is independent of x, and since $\mathrm{d}x = (1/a)\,\mathrm{d}y$, the above equation can be integrated to give

$$P = \frac{6\mu Q}{al}\left(\frac{1}{y^2}+C\right)$$

The constant of integration, C, can be found by putting $P = P_1$ at $y = y_1$

$$P = P_1 + \frac{6\mu Q}{al}\left(\frac{1}{y^2}-\frac{1}{y_1^2}\right)$$

The condition $P = P_2$ at $y = y_2$ gives either Q

$$\boxed{Q = \frac{\Delta P l y_1^2 y_2^2}{12\mu L\left(\dfrac{y_1+y_2}{2}\right)}} \tag{2}$$

or, by eliminating Q, the variation of P in the flow direction

$$\boxed{\frac{P-P_1}{P_2-P_1} = \frac{(1/y^2)-(1/y_1^2)}{(1/y_2^2)-(1/y_1^2)}} \tag{3}$$

Since $y = ax+b$, eqn. (3) can easily be expressed in terms of x.

3. A CONICAL PISTON IN A CYLINDRICAL SLEEVE

Let

L = length of piston
j_1 = radial clearance of upstream end (measured with piston central)
j_2 = radial clearance of downstream end (measured with piston central)
ϵ = eccentricity (assumed constant along length of piston).

Define the mean clearance by $j_m = (j_1+j_2)/2$ and a taper factor by $\Delta j = (j_2-j_1)/2$.

The preliminary analysis of the previous section may be applied to the angular section between the radial planes θ and $\theta+\mathrm{d}\theta$.

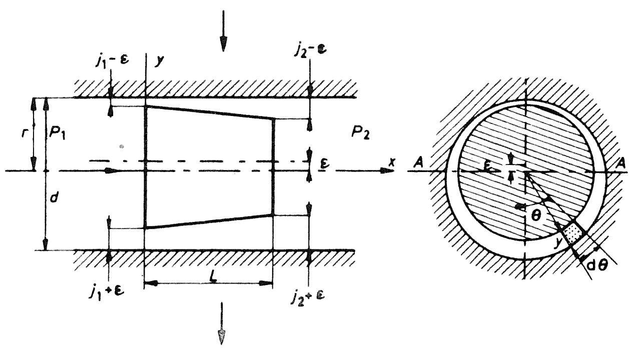


Figure 4.40

The radial force, f, exerted by the fluid on this section of the piston is

$$f = \int_{y_1}^{y_2} \mathrm{d}f$$

with $\mathrm{d}f = P\,\mathrm{d}S = Pr\,\mathrm{d}\theta\,\mathrm{d}x = (r\,\mathrm{d}\theta/a)P\,\mathrm{d}y$. Substituting the expression for P as a function of y, as determined above, gives

$$\boxed{f = Lr\,\mathrm{d}\theta\left(\frac{P_1 y_1 + P_2 y_2}{y_1 + y_2}\right)} \tag{4}$$

and, putting $(y_1+y_2)/2 = y_m$ and $(y_2-y_1)/2 = \Delta y$

$$\boxed{f = \frac{Lr\,\mathrm{d}\theta}{2}\left[(P_1+P_2)-(P_1-P_2)\frac{\Delta y}{y_m}\right]} \tag{5}$$

Three important conclusions can be drawn from these results:

(*i*) If the conicity is zero

$$\Delta y = 0 \qquad f = \frac{Lr\,\mathrm{d}\theta}{2}(P_1+P_2) \tag{6}$$

The force f is that corresponding to a linear decrease of pressure from P_1 to P_2. It is independent of θ, so that the resultant of all the forces f is zero. *A perfectly cylindrical valve will not stick.*

(*ii*) Consider a conical piston whose diameter decreases in the flow direction:

$$y_2 > y_1 \qquad \Delta y > 0$$

The force f on each section $\mathrm{d}\theta$ is less than it would be if the piston were cylindrical.

Since f decreases as $\Delta y/y_m$ increases, it is smaller on the side of the piston which is closer to the sleeve (the top in *Figure 4.40*) than on the opposite side. The resultant force therefore tends to aggravate the eccentricity and pushes the piston against the wall. *A conical valve whose diameter decreases in the flow direction may stick.*

(*iii*) Consider a conical piston whose diameter increases in the flow direction

$$\Delta y < 0$$

The force f is greater than that exerted on the cylindrical piston and increases as $\Delta y/y_m$ increases. It is therefore higher on the side of the piston which is closer to the sleeve. The resultant force tends to nullify any initial eccentricity. *A conical valve whose diameter increases in the flow direction will not stick and is self-centring.*

Let us now return to the analysis. It can be seen from *Figure 4.40* that

$$y_1(\theta) = j_1 + \epsilon\cos\theta$$

$$y_2(\theta) = j_2 + \epsilon\cos\theta$$

$$\therefore\ y_m = \frac{j_1+j_2}{2} + \epsilon\cos\theta = j_m + \epsilon\cos\theta$$

$$\Delta y = \Delta j$$

Eqn. (5) therefore becomes

$$f = \frac{Lr\,\mathrm{d}\theta}{2}\left[(P_1+P_2)-(P_1-P_2)\frac{\Delta j}{j_m+\epsilon\cos\theta}\right] \tag{7}$$

Due to symmetry the resultant is in the vertical plane (see *Figure 4.40*). It can be found by integrating $f\cos\theta$ between $\theta = 0$ and 2π. We know that the constant term has an integral of zero, so we have

$$F = \int_0^{2\pi} f\cos\theta = \frac{Lr}{2}(P_1-P_2)\frac{\Delta j}{j_m}\int_0^{2\pi}\frac{\cos\theta\,\mathrm{d}\theta}{1+(\epsilon/j_m)\cos\theta}$$

which gives

$$\boxed{F = \frac{\pi}{2}L\,\mathrm{d}(P_1-P_2)\frac{\Delta j}{\epsilon}\left[\frac{1}{\sqrt{1-(\epsilon/j_m)^2}}-1\right]} \tag{8}$$

In order to assess the order of magnitude of this force, it is convenient to relate it to a reference force, F_R, the force which would be obtained if all the upper half of the piston were subjected to a pressure P_2 and all the lower half to a pressure P_1, i.e.

$$F_R = L\,\mathrm{d}(P_1-P_2) \tag{9}$$

Defining a coefficient of sticking, α, by

$$\boxed{F = \alpha F_R} \tag{10}$$

gives the final result in the form

$$\boxed{\alpha = \frac{\pi}{2}\frac{\Delta j}{\epsilon}\left[\frac{1}{\sqrt{1-(\epsilon/j_m)^2}}-1\right]} \tag{11}$$

where

ϵ = eccentricity

j_m = radial clearance of the piston half-way along its length

$\Delta j = (j_1-j_2)/2$ = half difference of radial clearances at each end of the piston (*Figure 4.41*).

Consider now the case of limiting eccentricity, i.e. contact between piston and sleeve (*Figure 4.42*). In this case, $\epsilon = j_m-\Delta j$, and eqn. (11) becomes

$$\alpha = \frac{\pi}{2}\frac{\Delta j}{(j_m-\Delta j)}\left[\frac{1}{\sqrt{1-(j_m-\Delta j/j_m)^2}}-1\right]$$

Putting

$$\boxed{\tau = \frac{\Delta j}{j_m}} \tag{12}$$

this becomes

$$\boxed{\alpha = \frac{\pi}{2}\frac{\tau}{(1-\tau)}\left[\frac{1}{\sqrt{1-(1-\tau)^2}}-1\right]} \tag{13}$$

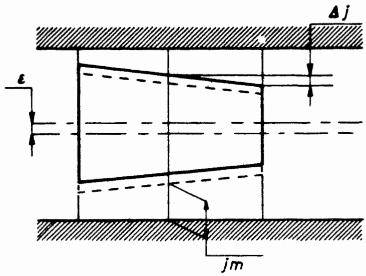


Figure 4.41

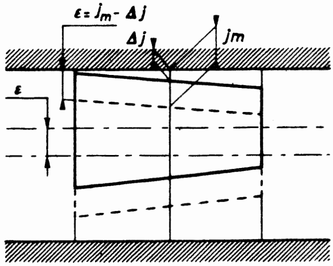


Figure 4.42

Note—Manufacturers' inspection departments normally measure the mean diametral clearance, $J_m = 2j_m$ and the difference in the diameters of the two ends, $g = 4\Delta j$. A more practical definition of τ is therefore

$$\boxed{\tau = \frac{1}{2}\frac{g}{J_m}}$$

Figure 4.43 shows the variation of α with τ. It can be seen that the coefficient α has a maximum value of about 0·27—a value which may well occur in practice.

The lateral force due to the conicity of the piston may be more than a quarter of the reference force.

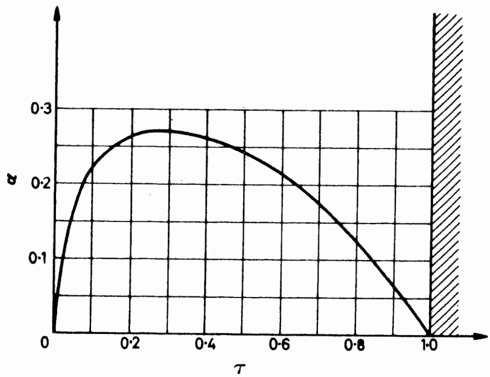


Figure 4.43. Variation of α with τ

For $\tau = 0$, $\alpha = 0$ since conicity is zero; for $\tau = 1$, $\alpha = 0$ for the same reason, i.e. the upstream diameter of the piston is exactly equal to the sleeve diameter.

4. NUMERICAL EXAMPLE

Consider a valve with the following dimensions:

nominal diameter	8 mm
piston length	10 mm
pressure difference	200 kg/cm^2
mean diametral clearance (sleeve diameter − piston diameter)	$J_m = 5\ \mu$
piston taper (inlet diameter − outlet diameter)	$g = 1\ \mu$

This gives the following results
Reference force

$$F_R = \Delta PL\,d = 160\text{ kg}$$
$$\tau = \tfrac{1}{2}\cdot\tfrac{1}{5} = 0{\cdot}1$$

From *Figure 4.43* the corresponding value of α is 0·225, so $F = 36$ kg.

If the coefficient of friction is taken as 0·15, the force resisting longitudinal motion is 5·4 kg.

5. PRECAUTIONARY MEASURES

The form of *Figure 4.43* indicates that the alleviation of sticking *cannot* be assured *either by increasing the clearance or by decreasing the taper of the piston.* One effective method would be to change the direction of the taper, but this is of no use when the flow may take place in either direction. *The most common method is to groove the piston, in order to reduce the pressure.*

The effectiveness of the grooves may be estimated by a calculation which is lengthy but not difficult. The results are shown qualitatively in *Figure 4.44* which gives the pressure distributions, $P = f(x)$, for $\theta = 0$ and $\theta = 180°$ for two identical valves, one plain and one grooved.

The lateral force is proportional to the area between the two curves (the shaded areas in *Figure 4.44*), so the effectiveness of the grooves can be clearly seen. The presence of the groove allows the pressure to be equalized around the circumference, so that the two curves coincide at each groove.

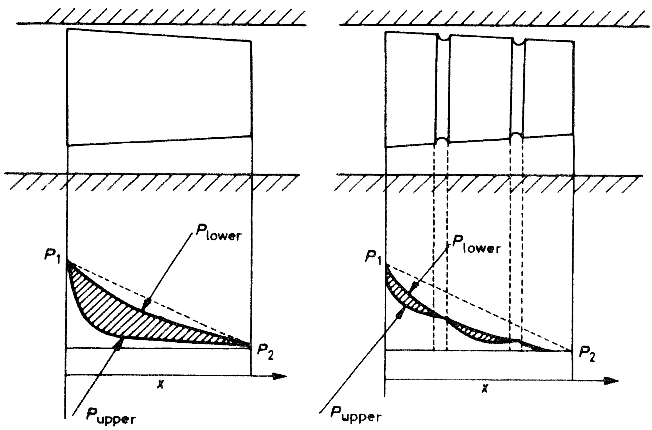


Figure 4.44

Sweeney has shown that the factor α is divided by

2·5 for 1 groove
16 for 3 grooves
37 for 7 grooves.

These results are for grooves symmetrically disposed along the length of the piston. But it can be shown that it is more effective to place the grooves at the upstream end.

This problem is now the subject of a certain amount of literature. In particular, *Fluid Power Control*[1] gives an excellent analysis of the problem, including a qualitative analysis of less simple shapes than the cone, together with a consideration of non-parallel eccentricity, a comparison of theoretical and experimental results, and an estimate of the time constant necessary to establish and overcome valve sticking.

REFERENCES

[1] Blackburn, Reethof and Shearer *Fluid Power Control*, New York (Wiley) 1960
[2] From Alfred Teves, Maschinen-und Armaturenfabrik KG, Frankfurt, French Patent No. 1,245,197
[3] From Guillon, M. *Apparecchiature idrauliche e pneumatiche* **4** (1964) No. 21 (14, Via Fantoni, Milan, Italy)

PART II. DYNAMIC PERFORMANCE

In the past, the analysis of hydraulic systems was limited to the determination of the steady flow conditions, as discussed in Part I. Before building a prototype system, however, it is essential to determine whether these steady flow conditions are *physically possible*, i.e. if they are *stable* conditions.

It is very often equally necessary to know the way in which the steady conditions are established or how they are changed from one state to another, i.e. *the transient regimes* (response speed, damping). For this reason, it is necessary to examine not only the parameters and variables of the system, but their *successive derivatives with respect to time*, i.e. to *derive* and *use* the *differential equations* defining the operation of the system under consideration.

The form of these differential equations has been established for some time, but their solution was possible only in a few very simple cases. In the last few decades, however, very elegant methods of solution developed in the field of transmissions have assisted in solving these equations and, in particular, have permitted most of the required information to be acquired *without actually solving the equations.*

Contrary to widespread opinion, these methods are very efficient in hydraulics, and the following Chapters have been written in order to demonstrate this.

5

FORMING THE EQUATIONS

5.1. EQUATIONS OF HYDRAULIC SYSTEMS

5.1.1. GENERAL CONSIDERATIONS

Due to the existence of modern methods of analysis and the various types of computers now available, *the main problem in the analysis of hydraulic systems consists of establishing the relevant equations.*

There is no procedure or machine which can automatically form the equations. It must also be appreciated that the use of analogical methods does not change this problem. At best, it eases the dynamic analysis by identifying the system of differential equations obtained with another system which has already been analysed or just with one whose symbols are more familiar, thus making the handling easier (see Appendix 5.1).

5.1.2. THE NATURE OF THE EQUATIONS

Forming an equation for a hydraulic system is obviously the same as for any other type of system. With the exception of geometric and kinematic relationships, the equations reduce to either equations of force (or of torque) or to equations of flow.

In general, these equations are not linear*. They can usually be *made linear*, however, near a point, so that they are amenable to the simplest and most useful methods of analysis.

Finally, it must be noted that the analysis is greatly simplified by relating all values of a variable A to its value A_0 at the steady-state condition under consideration. The variable then becomes $\Delta A = A - A_0$. This obviously eliminates the constant terms of the differential equations. This method will be systematically adopted here.

5.2. EQUATIONS OF FORCE (OR OF TORQUE)

5.2.1. GENERAL CASE

These equations express the equilibrium of *components in motion* under the action of the forces (or torques) to which they are subjected, i.e. the forces†

* The definition of a 'linear equation' is that used by physicists, a linear differential equation with *constant coefficients*. This is more restrictive than the mathematical definition.

† In the following pages the term 'force' should be understood to mean 'force or torque'.

which are a function of the position, x (or θ), of the component considered and of the successive derivatives of x (or θ)

$$F_i(x) \qquad \text{[or the torque } C_i(\theta)\text{]}$$

and/or the forces which are a function of external parameters (controls, perturbations, etc.) or of certain system variables (pressure, flow, rotational speed, etc.)

$$F_i(e) \qquad \text{[or } C_i(e)\text{]}$$

Applying the fundamental law of dynamics, we have

$$\Sigma F_i(x) + \Sigma F_i(e) = M \frac{d^2x}{dt^2} \tag{1}$$

or

$$\Sigma C_i(\theta) + \Sigma C_i(e) = I \frac{d^2\theta}{dt^2} \tag{1'}$$

where M and I are the mass and the moment of inertia of the component, respectively.

5.2.2. LINEAR EQUATIONS

If the forces which depend on x can be written

$$F_i(x) = F_{0_i} + R_i x - f\frac{dx}{dt}$$

(forces which are a linear function of x and viscous friction forces) and if the forces which depend on external parameters can be written

$$F_i(e) = F'_{0_i} + Ke$$

the force equation becomes

$$\Sigma F_{0_i} + \Sigma F'_{0_i} + \Sigma(Ke) + x\Sigma R_i - \frac{dx}{dt}\Sigma f = M\frac{d^2x}{dt^2}$$

If we put

$$X = x + \frac{\Sigma F_{0_i} + \Sigma F'_{0_i}}{\Sigma R_i}$$

the origin becomes the position of static equilibrium, so that

$$\Sigma(Ke) = -\Sigma R_i X + \Sigma f \frac{dX}{dt} + M\frac{d^2X}{dt^2}$$

and finally replacing each stiffness R_i by the corresponding restoring stiffness $r_i = -R_i$ (see Section 4.5)

$$\boxed{\Sigma(Ke) = \Sigma r_i X + \Sigma f \frac{dX}{dt} + M\frac{d^2X}{dt^2}} \tag{2}$$

The equation for torque has the same general form:

$$\Sigma\,(K'e) = \Sigma C_i\,\Theta + \Sigma\,\varphi\,\frac{d\Theta}{dt} + I\,\frac{d^2\Theta}{dt^2} \tag{2'}$$

Note—In this equation it can be seen that the restoring stiffness, $r = -R$, defined in Section 4.5 has a certain mnemonic interest in the formation of the equations:

the inertia forces have the *opposite sign* to d^2X/dt^2
the friction forces have the *opposite sign* to dX/dt
the restoring forces have the *opposite sign* to X

so that they all appear on the right-hand side of the equation with positive signs.

For an example of the equation for the flapper plate of a flapper-and-nozzle valve, see Section 5.5.1.

5.3. FLOW EQUATIONS

5.3.1. THE FORM OF THE FLOW EQUATION

Flow equations express the law of the *conservation of mass*. They are normally applied to components, or parts of components, in which the *pressure* can be regarded as being approximately *uniform*.

The equation is formed by equating the difference between the mass flow entering a system in a given time and that leaving it, to the variation of mass within the system.

The masses entering and leaving the system per unit time are called the inlet mass flow, m_1, and the outlet mass flow, m_2. If the mass of fluid contained in the system is M, the volume V and the density of the fluid ρ, the flow equation is

$$m_1 - m_2 = \frac{dM}{dt} \tag{3}$$

or, since $M = \rho V$

$$\boxed{m_1 - m_2 = \rho\frac{dV}{dt} + V\frac{d\rho}{dt}} \tag{4}$$

This is the most general form of the flow equation. We will develop it to a form which is easier to use and then discuss each of its terms.

The volume flow, Q, is a more practical parameter than the mass flow, m, and is related to it by the equation

$$m = \rho Q$$

Eqn. (4) now becomes

$$\boxed{Q_1 - Q_2 = \frac{dV}{dt} + \frac{V}{\rho}\frac{d\rho}{dt}} \tag{5}$$

This equation simply states that the difference between the inlet flow and the outlet flow is equal to the change in volume of the system plus the small volume of fluid due to the compression or expansion of the fluid initially in the system (which corresponds to a fraction $\Delta\rho/\rho$ of the initial volume V).

Some authors state, in a way which is more eloquent than rigorous:

Inlet flow − outlet flow = deformation flow + compressibility flow

We shall consider each of these flows in turn.

5.3.2. THE COMPRESSIBILITY FLOW, $V/\rho \, . \, \mathrm{d}\rho/\mathrm{d}t$

The liquids used in hydraulic systems are only approximately incompressible, in comparison with a gas, for example. In fact, their compressibility, although small, is the source of almost all the instabilities found in hydraulic systems.

Although the density changes are sufficiently small to allow ρ to be replaced by a constant mean value, ρ_0, they occur so rapidly that the term $V/\rho.\mathrm{d}\rho/\mathrm{d}t$, i.e. the 'compressibility flow', is by no means negligible in comparison with the other flows considered.

The compressibility of a liquid at a given temperature is characterized by the variation of its density with pressure which may take the form shown in *Figure 5.1*.

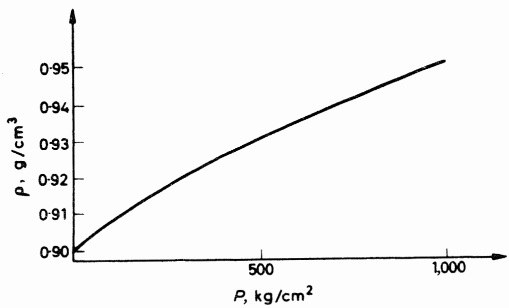


Figure 5.1

For ease of analysis, it is desirable to represent this curve by an approximate algebraic expression, linear if possible. An expression of the form $\Delta\rho/\rho = K(\Delta P/P)$, similar to that used for gases, where $K = 1$ for isothermal processes and $K = 1/\gamma$ for adiabatic processes, is obviously not suitable. An equation of the form $\Delta\rho = K\,\Delta P$ is much better.

In practice, we use

$$\frac{\Delta\rho}{\rho} = \mu\,\Delta P \tag{6}$$

or, more often

$$\boxed{\frac{\Delta\rho}{\rho} = \frac{\Delta P}{B}} \tag{6'}$$

where μ = coefficient of compressibility, B = bulk modulus. These are virtually the same as the expression suggested above, since ρ is practically constant.

Eqn. (6) and (6′) therefore give values of μ and B valid for a certain temperature and in the neighbourhood of a certain pressure.

By eliminating the density, ρ, between eqn. (6′), defining the bulk modulus*, B, and the expression for the compressibility volume, $V/\rho \,.\, \mathrm{d}\rho/\mathrm{d}t$, we obtain the new term

$$\boxed{\frac{V}{B}\frac{\mathrm{d}P}{\mathrm{d}t}} \tag{7}$$

whose magnitude is proportional to the rate of change of pressure†.

The bulk modulus, B, of normal hydraulic fluids at normal temperatures is in the neighbourhood of *15,000 bars.*

It *decreases* rapidly with temperature (approximately 1 per cent per 2°C at 100°C for normal fluids).

It *increases* slightly with pressure (approximately 1 per cent per 20 bars at 200 bars for normal fluids).

It is *appreciably less* for special hydraulic fluids used for high-temperature operation.

It decreases very rapidly when air is entrained in the oil. This is shown by the following calculation.

Suppose a volume, V, initially at pressure P contains a volume of air, $V_a = \epsilon V$. The volume of oil is therefore $V_0 = (1-\epsilon)V$. An increase of pressure, ΔP, causes a decrease of volume

$$\Delta V = \Delta V_0 + \Delta V_a = \frac{V_0}{B}\Delta P + \frac{V_a}{P}\Delta P$$

If ϵ is small, which is generally true, $V_0 \simeq V$, so that we have

$$\Delta V = V\left(\frac{1}{B}+\frac{\epsilon}{P}\right)\Delta P$$

and

$$\boxed{\frac{\mathrm{d}V}{\mathrm{d}t} = V\left(\frac{1}{B}+\frac{\epsilon}{P}\right)\frac{\mathrm{d}P}{\mathrm{d}t}}$$

This equation shows that when a relative volume, ϵ, of air is mixed with the oil (the volume obviously measured at pressure P), the bulk modulus B must be replaced by B', defined by

$$\boxed{\frac{1}{B'} = \frac{1}{B}+\frac{\epsilon}{P}}$$

* The bulk modulus, B, is chosen in preference to μ because it has the same dimensions as pressure, and as a result the numerical valves are easier to understand and remember.

† This result is due to the choice of a linear equation defining B.

B' may be considered as constant provided that P does not vary considerably.

For a general idea of the effect of air entrainment, note that if $B = 15{,}000$ bars and $P = 150$ bars
with

$$\epsilon = \frac{1}{1{,}000}, \qquad B' = 0{\cdot}9B$$

and

$$\epsilon = \frac{1}{100}, \qquad B' = 0{\cdot}5B$$

(see also *Figure 5.2*).

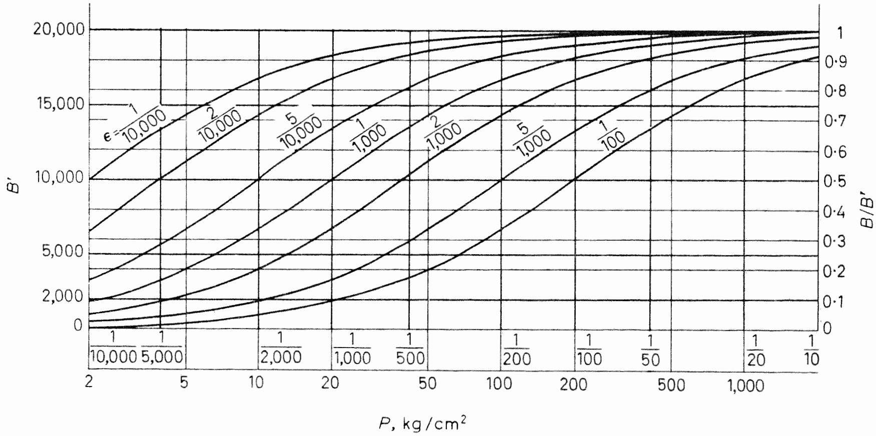


Figure 5.2

B = bulk modulus without air; B' = bulk modulus with air; ϵ = volumetric proportion of air; P = pressure. The second scale gives B' directly as a function of P and ϵ for $B = 20{,}000$ kg/cm^2.

Note 1—If we use eqn. (6′) to define a bulk modulus B for a gas during an isothermal process in the same way as we have done for liquids, we get $B = P$. This leads to two important results:

(*i*) at normal operational pressures, of the order of 150 bars, hydraulic fluids are about 100 times less compressible than gases ($B = 15{,}000$ as against 150);

(*ii*) since, for a given application, the effects of compressibility on gases are independent of the operational pressure, we can expect that for fluids, whose bulk modulus remains practically constant instead of increasing like P, these effects will be aggravated when the operational pressure is increased.

This result is confirmed by calculation and practical experience.

Note 2—Since density is defined by $\rho = M/V$, we have for a given mass of gas

$$\frac{\mathrm{d}V}{V} = -\frac{\mathrm{d}\rho}{\rho}$$

which when substituted in eqn. (6′) gives

$$\frac{\Delta V}{V} = -\frac{\Delta P}{B} \tag{8}$$

This is often taken as the equation defining the bulk modulus. It states that the relative decrease in volume of a compressed fluid is equal to the absolute increase in pressure divided by the bulk modulus, B.

NUMERICAL VALUES OF THE BULK MODULUS, B

Authors of reports on hydraulic fluids are usually very reticent about quoting values of the bulk modulus. The data given are often incomplete, temperature or pressure being omitted, sometimes they are even contradictory. The figures given below are mean values.

(*a*) *Mean values between* 0 *and* 200 kg/cm^2 *for different fluids at* 25°C

Water	23,500
Skydrol 7000	23,000
Skydrol 500	21,500
Standard hydraulic fluid*	17,000

* Standard hydraulic fluid is defined by the standards: British DTD 585; American MIL-H-5606; French AIR 3520. It is also known by commercial names, the best known of which are Aeroshell 4, FHS 1 and Univis J 43.

(*b*) *Mean values between* 0 *and* 200 kg/cm^2 *for standard hydraulic fluid, as a function of temperature*

t, °C	*Bulk modulus, B*
−50	22,000
0	19,000
50	15,500
100	12,500
150	9,500

(*c*) *Mean values between* 0 *and* P kg/cm^2 *for* FH 8 (*high-temperature*) *fluid at* 40°C, *as a function of* P

P, kg/cm^2	*Bulk modulus, B*
100	11,600
200	12,400
300	13,000

(*d*) *Fuels* (*kerosene*): between 12,000 and 15,000.

5.3.3. THE DEFORMATION FLOW, $(\mathrm{d}V)/(\mathrm{d}t)$

There are two types of deformation flow: those which are functional, such as in the displacement of a piston, and those which are incidental and secondary.

The first may usually be expressed without difficulty as a function of a *parameter of position*. Thus, for a ram of effective area S (*Figure 5.3*) in which

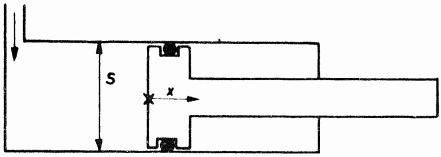


Figure 5.3

the position of the piston is defined by x, the change in volume is $\Delta V = S\,\Delta x$, and the deformation flow is given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = S\frac{\mathrm{d}x}{\mathrm{d}t}$$

The second type of deformation flow is due to the distortion of the materials constituting the walls of the system.

In the elastic region, assuming the linear relationship $\Delta V = k\,\Delta P$, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} = k\frac{\mathrm{d}P}{\mathrm{d}t}$$

This expression has the same form as that for compressibility flow, eqn. (7), and the two terms are often grouped together:

$$\frac{V}{B}\frac{\mathrm{d}P}{\mathrm{d}t} + k\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{V}{B'}\frac{\mathrm{d}P}{\mathrm{d}t}$$

where B', the *apparent* bulk modulus of the liquid in the system, is defined by

$$\frac{1}{B'} = \frac{1}{B} + \frac{k}{V} \tag{9}$$

The quality of the casing design of a component may therefore be evaluated by comparing $1/B$ and k/V.

Example—Consider a thin cylindrical casing with rigid ends of diameter D, thickness e and length L, filled with a liquid at pressure P which may reach a maximum value, P_M. The tensile stress in the container perpendicular to the axis of the cylinder is $\sigma_1 = PD/2e$, and the stress parallel with the axis, $\sigma_2 = PD/4e = \frac{1}{2}\sigma_1$.

If σ_{1_M} is the maximum value of σ_1 corresponding to $P = P_M$, a variation $\mathrm{d}P$ of pressure causes the variations $\mathrm{d}\sigma_1 = \sigma_{1_M}\,\mathrm{d}P/P_M$ and $\mathrm{d}\sigma_2 = \frac{1}{2}\sigma_{1_M}\,\mathrm{d}P/P_M$.

The relative variation of the diameter of the cylinder is therefore

$$\frac{\mathrm{d}D}{D} = \frac{\mathrm{d}\sigma_1}{E} = \frac{\sigma_{1M}}{EP_M}\mathrm{d}P$$

and the relative variation of length

$$\frac{dL}{L} = \frac{d\sigma_1}{2E} = \frac{\sigma_{1_M}}{2EP_M} dP$$

Hence, the total relative variation in volume is

$$\frac{dV}{V} = 2\frac{dD}{D} + \frac{dL}{L} = \frac{5\sigma_{1_M}}{2EP_M} dP$$

and the deformation flow

$$\frac{dV}{dt} = \frac{5V\sigma_{1_M}}{2EP_M}\frac{dP}{dt}$$

so that the coefficient k/V, defined by eqn. (9), is

$$\frac{k}{V} = \frac{5\sigma_{1_M}}{2EP_M}$$

If the cylinder is made of aluminium alloy ($E = 8{,}000$ kg/mm^2) and has been designed for a maximum tangential tensile stress σ_{1_M} of 10 kg/mm^2 under a maximum operational pressure of 200 bars

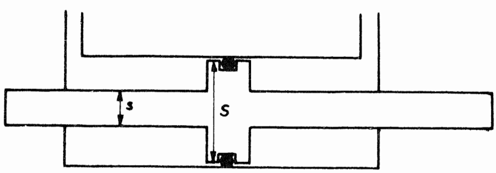
$$\frac{k}{V} = \frac{1}{64{,}000} \text{ cm}^2/\text{kg}$$

Thus, with $B = 15{,}000$ kg/cm^2

$$B' = 12{,}000 \text{ kg/cm}^2$$

The elasticity of the casing has the same effect as a 20 per cent reduction in the bulk modulus of the liquid.

Figure 5.4



Now consider the equal-area ram shown in *Figure 5.4* in which the ratio of the cross-sectional area of the piston rod, s, to that of the piston, S, is $\lambda = s/S$. Since the volume of oil is now $(1-\lambda)$ times that of the plain cylinder, if we neglect the deformation of the piston rod itself, the variation in volume becomes

$$\frac{dV}{V} = \frac{5}{2}\frac{1}{(1-\lambda)}\frac{\sigma_{1_M}}{EP_M} dP$$

wherefrom

$$\frac{k}{V} = \frac{5}{2}\frac{1}{(1-\lambda)}\frac{\sigma_{1_M}}{EP_M}$$

With the numerical values given above and with, say, $\lambda = 0{\cdot}5$

$$\frac{k}{V} = \frac{1}{32{,}000}\frac{\text{cm}^2}{\text{kg}}$$

$$B' \triangleq 10{,}000 \text{ kg/cm}^2$$

The elasticity of the casing cannot be ignored. Its effect becomes even more significant, if the maximum stress (or, more precisely, the ratio σ_M/E) is increased; if the shapes are not simple (as shown in the example of a ram of annular cross section); finally, and most important, if the structure contains very elastic parts such as diaphragms, capsules or seals.

For example, flexible tubes designed for pressures of the order of 200 kg/cm^2 undergo an increase in volume of about 20 per cent as the pressure is increased from 0 to 200 kg/cm^2, which corresponds to an apparent bulk modulus, B', of 1,000 kg/cm^2.

If the casing is much more elastic than the liquid:

$$\frac{1}{B'} = \frac{1}{B} + \frac{1}{B''} \triangleq \frac{1}{B''}$$

Thus, if a system is made up of an enclosure of volume V_1, whose apparent total bulk modulus is B', together with a flexible pipe of volume V_2, whose apparent bulk modulus is B'', starting from the equation of definition it can easily be shown that the overall apparent bulk modulus for the whole system, B_0, is

$$\boxed{\frac{1}{B_0} = \frac{V_1}{(V_1 + V_2)}\frac{1}{B'} + \frac{V_2}{(V_1 + V_2)}\frac{1}{B''}}$$

Example—If a flexible pipe of volume $V_2 = \frac{1}{10}V_1$ is added to the ram considered above, for which we obtained $B' = 10{,}000$ kg/cm^2, the overall bulk modulus becomes

$$B_0 \triangleq 5{,}500 \text{ kg/cm}^2$$

These considerations preclude the use of flexible tubing between the valve and ram chambers in high-performance hydraulic servo systems.

5.3.4. INLET AND OUTLET FLOWS, Q_1 AND Q_2

Let E be the enclosure or chamber in the component considered and let the pressure inside it be P. E may be connected to a certain number of other enclosures, E_i, at pressures P_i by means of passages or channels C_i which may contain restricting orifices (valves, fixed orifices, etc.).

The flows Q_i passing from E_i to E or from E to E_i depend essentially on the values of P and P_i, the connections C_i and the characteristics of the fluid, F.

If neither P nor P_i nor the pressure at any point in the connecting passages C_i fall below a value low enough to cause vaporization or cavitation, the flow, Q, depends only on the difference in the pressures P and P_i and not on their absolute values.

The problem is now to find the variation of Q_i with $(P_i - P)$, C_i, and F. It is obvious that this is the same problem as that of finding the pressure drop in the flow Q_i while passing through the connections C_i, i.e. the problem of determining the variation of $(P_i - P)$ with Q_i, C_i and F. This is the problem of the determination of loss of head which is covered in Chapter 1; for an example see Section 5.5.1 below.

5.4. LINEARIZATION

5.4.1. THE PROBLEM OF LINEARIZATION

As we have seen, most hydraulic equations are not linear.

The most interesting and simple methods of dynamic analysis are those which have been developed *for linear equations*. Every possible means must therefore be tried to make the equations linear.

This linearization often presents no problem. It can be done in the neighbourhood of a particular point by considering the curve as *equivalent to its tangent at that point* (see below).

Sometimes this operation is not possible, in particular when the slope of the tangent is zero, infinite or discontinuous.

In this case, however, on certain supplementary conditions, it is possible to replace the non-linear equation by an 'equivalent' linear one (see Section 5.4.3).

5.4.2. STANDARD RULES OF LINEARIZATION

5.4.2.1. *Linearization of a function of a variable, $y = f(x)$ in the neighbourhood of the point (x_0, y_0)*

Provided that the slope $\mathrm{d}y/\mathrm{d}x$ of the curve $y = f(x)$ at the point (x_0, y_0) is finite, we can replace $y = f(x)$ by $\Delta y = (\mathrm{d}y/\mathrm{d}x)_0 \, \Delta x$ near the point.

If $y = x^n$

$$\Delta y = n \frac{y_0}{x_0} \Delta x \tag{10}$$

and in terms of the *reduced variables*, $\Delta Y = \Delta y / y_0$ and $\Delta X = \Delta x / x_0$

$$\Delta Y = n \, \Delta X \tag{11}$$

If $y = f(x)$ is known experimentally, it is advisable to plot the curve $y = f(x)$ in logarithmic coordinates and to measure the slope, K, of the tangent at point (x_0, y_0)

$$K = \frac{\mathrm{d}\,(\log y)}{\mathrm{d}\,(\log x)} = \frac{\mathrm{d}y / y_0}{\mathrm{d}x / x_0} = \frac{\mathrm{d}Y}{\mathrm{d}X}$$

so that

$$\Delta Y = K \, \Delta X$$

This method is particularly useful, since it generally precludes errors of units or scales which are extremely common in experimental curves. K is a convenient number to deal with: it is rarely smaller than 0·2 or larger than 5. This simply means that the algebraic curve $y = x^n$ tangential to the point (x_0, y_0) on the experimental curve, usually has an index lying between 0·2 and 5.

5.4.2.2. *Linearization of a function with several variables*

Linearization of a sum

$$z = ax^n + \beta y^p = z_1 + z_2$$

$$\frac{dz}{z_0} = \frac{dz_1}{z_{10}}\frac{z_{10}}{z_0} + \frac{dz_2}{z_{20}}\frac{z_{20}}{z_0} = \frac{dx}{x_0}\frac{nax_0^n}{(ax^n + \beta y^p)} + \frac{dy}{y_0}\frac{p\beta y_0^p}{(ax^n + \beta y^p)}$$

$$= \frac{dz_1}{z_{10} + z_{20}} + \frac{dz_2}{z_{10} + z_{20}} \tag{12}$$

Linearization of a product

$$z = x^n y^p = z_1 z_2$$

With the exception of points near the origin

$$\frac{dz}{z_0} = n\frac{dx}{x_0} + p\frac{dy}{y_0} \tag{13}$$

Linearization of a power

$$y = a^x$$

$$dy = a^x \text{ Log } a \, dx$$

$$\frac{dy}{y_0} = \text{Log } a \, dx$$

An example for linearization of the equations for a hydraulic potentiometer is given in Section 5.5.1.

5.5. EXAMPLES OF THE DERIVATION OF EQUATIONS AND OF LINEARIZATION

5.5.1. THE HYDRAULIC POTENTIOMETER WITH FLAPPER-AND-NOZZLE VALVE

The hydraulic potentiometer is an element which is frequently used in hydraulic components. Together with a proportional electric control, it constitutes the first hydraulic stage of most electrohydraulic servo valves and is itself the basis of most hydraulic servo systems. The analysis of this element is therefore fundamental in a consideration of such systems.

The layout is shown diagrammatically in *Figure 5.5* which also defines the notation used in the following analysis.

The torque equation—The flapper plate, A, which rotates about an axis O with a moment of inertia I, is acted upon by:

a command torque, C_c, proportional to the current, i, applied to the control windings

$$C_c = Kil_e$$

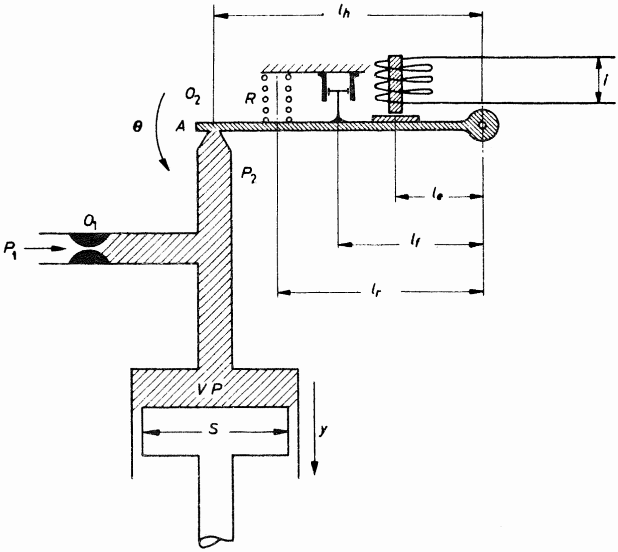


Figure 5.5

a hydraulic couple, C_h, proportional to the pressure difference, $P-P_2$ and the area, σ, of the variable orifice O_2

$$C_h = -\sigma(P-P_2)l_h$$

a restoring couple from the spring, R, of stiffness r

$$C_r = (R_0 - rl_r\theta)l_r$$

a friction couple due to the dash-pot D of coefficient of viscous friction f

$$C_f = -fl_f\frac{d\theta}{dt}l_f$$

an inertia torque

$$C_I = -I\frac{d^2\theta}{dt^2}$$

The torque equation is therefore:

$$l_eKi - l_h\sigma(P-P_2) + l_r(R_0 - l_r r\theta) - l_f^2 f\frac{d\theta}{dt} - I\frac{d^2\theta}{dt^2} = 0$$

But for static equilibrium

$$l_eKi_0 - l_h\sigma(P_0 - P_{2_0}) + l_r(R_0 - l_r r\theta_0) = 0$$

so that, putting

$$\Delta i = i - i_0$$

$$\Delta P = P - P_0$$

$$\Delta\theta = \theta - \theta_0$$

and assuming that $P_2 = P_{2_0}$, which is generally true, we obtain by subtraction

$$\boxed{l_e K\,\Delta i - l_h \sigma\,\Delta P = l_r^2 r\,\Delta\theta + l_f^2 f\frac{\mathrm{d}\,\Delta\theta}{\mathrm{d}t} + I\frac{\mathrm{d}^2\Delta\theta}{\mathrm{d}t^2}} \tag{14}$$

The flow equation—Let

Q_1 = flow through the *fixed* orifice O_1
Q_2 = outlet flow through the *variable* orifice O_2
V = volume under the variable pressure P (shaded in *Figure 5.5*)
S = effective area of piston
y = position of piston
B = bulk modulus of the oil

Neglecting any deformation of the body of the ram, the flow equation has been shown above to be

$$\boxed{Q_1 - Q_2 = S\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{V}{B}\frac{\mathrm{d}P}{\mathrm{d}t}}$$

Since the flow in the two orifices is generally turbulent, we have*

$$Q_1 = K_1\sigma_1\sqrt{P_1 - P}$$
$$Q_2 = K_2\sigma_2\sqrt{P - P_2}$$

The equation for Q_1, expressed in logarithmic form and differentiated, becomes

$$\frac{\Delta Q_1}{Q_{1_0}} = \frac{\Delta\sigma_1}{\sigma_{1_0}} + \frac{1}{2}\frac{\Delta(P_1 - P)}{(P_1 - P)_0}$$

But since σ_1 is fixed and P_1 is generally constant

$$\frac{\Delta Q_1}{Q_{1_0}} = -\frac{1}{2}\frac{\Delta P}{(P_1 - P_0)}$$

$$Q_1 = Q_{1_0}\left[1 - \frac{1}{2}\frac{\Delta P}{(P_1 - P_0)}\right]$$

For Q_2, the variation of σ_2 must be taken into account. For small displacements, σ_2 will be proportional to θ.

With the sign conventions shown in *Figure 5.5*

$$\frac{\Delta\sigma_2}{\sigma_{2_0}} = -\frac{\Delta\theta}{\theta_0}$$

so that the equation for Q_2, derived like that for Q_1, becomes

* The area σ of the orifice O_2 should not be confused with the area σ_2 which is that through which the fluid flows between the orifice O_2 and the plate: σ is a circle while σ_2 is collar-shaped.

$$Q_2 = Q_{2_0}\left[1-\frac{\Delta\theta}{\theta_0}+\frac{1}{2}\,\frac{\Delta P}{(P_0-P_2)}\right]$$

Substituting the expressions for Q_1 and Q_2 in the flow equation, together with the steady-state result, $Q_{1_0} = Q_{2_0}-Q_0$, gives

$$\boxed{\frac{Q_0}{2}\left[\frac{1}{P_1-P_0}-\frac{1}{P_0-P_2}\right]\Delta P+\frac{V}{B}\,\frac{\mathrm{d}\,\Delta P}{\mathrm{d}t}+S\frac{\mathrm{d}\,\Delta y}{\mathrm{d}t}-Q_0\frac{\Delta\theta}{\theta_0} = 0} \tag{15}$$

The force equation, (14), contains the variables Δi, ΔP and $\Delta\theta$. The flow equation, (15), contains the variables ΔP, Δy and $\Delta\theta$ (and their successive derivatives). Elimination of the internal variable, $\Delta\theta$, gives an expression relating Δi, ΔP and Δy. Using the operational form (see Chapter 6) and putting

$$l_r^2+l_f^2fp+Ip^2 = \mu(p)$$

and

$$\frac{1}{2}\left[\frac{1}{P_1-P_0}+\frac{1}{P_0-P_2}\right] = \lambda$$

eqns. (14) and (15) become

$$\begin{cases} l_eK\,\Delta i-l_h\sigma\,\Delta P = \mu(p)\,\Delta\theta \\ \left(\lambda Q_0+\dfrac{V}{B}p\right)\Delta P+Sp\,\Delta y = \dfrac{Q_0}{\theta_0}\,\Delta\theta \end{cases}$$

giving

$$\boxed{\left\{Q_0\left[\lambda+\frac{l_h\sigma}{\theta_0\mu(p)}\right]+\frac{V}{B}p\right\}\Delta P+Sp\,\Delta y = \frac{Q_0l_eK}{\theta_0\mu(p)}\,\Delta i} \tag{16}$$

The valve shown in *Figure 5.5* thus yields a relationship between the *input*, Δi, and two *outputs*, ΔP and Δy. In order to determine ΔP and Δy, knowing Δi and the various parameters of the system, it is also necessary to know the downstream impedance, i.e. the equation between ΔP and Δy for the receiving component situated downstream of the potentiometer, for which no assumption has yet been made.

If this receiver is of the constant-pressure type, ΔP is effectively zero and the output Δy may be found by putting $\Delta P = 0$ in eqn. (16). Unfortunately, in practice the impedance of receivers is seldom zero or infinity, and their consideration as such gives only a rough estimate of the performance.

5.5.2. THE PUMP-MOTOR SERVO. EXAMPLE OF THE ANALYSIS OF A HYDRAULIC SERVO SYSTEM

Consider the classical hydraulic servomotor system consisting of a variable displacement pump coupled to a fixed displacement hydraulic motor. The output from the pump is connected to the inlet of the motor so that, if the system is perfect, the ratio of the rotational speeds of the pump and the motor is equal to the inverse ratio of their displacement volumes

$$\frac{N_m}{N_p} = \frac{d_p}{d_m}$$

If the pump is driven by a constant-speed power source, the speed of the motor is controlled by the lever varying the pump displacement. This system is a servo system in which the input is the position of the pump displacement lever and the output is the motor speed. Let

D_p = maximum displacement volume of the pump
x = relative position of the pump displacement lever ($x = d_p/D_p$)
V = volume of fluid in the lines between outlet of pump and inlet of motor
P = delivery pressure from the pump
B = bulk modulus of the oil
J = moment of inertia of the load applied to motor shaft
C_R = resisting torque applied to motor shaft (assumed proportional to rotational speed of motor, $C_R = RN_m$)
Q_p = flow through pump
Q_m = flow through motor
Q_c = compressibility flow
Q_f = internal leakage flow, defined by the leakage coefficient, F: $Q_f = F \,.\, P$.

Note that speeds are expressed in radians per second and displacement volumes in radians, not in revolutions.

The flow equations are

$$Q_p = Q_m + Q_c + Q_f$$

$$Q_p = N_p x Dp$$

$$Q_m = N_m d_m$$

$$Q_c = \frac{V}{B}\frac{\mathrm{d}P}{\mathrm{d}t}$$

$$Q_f = FP$$

By eliminating Q_p, Q_m, Q_c and Q_f, we have the single flow equation

$$\boxed{xN_pD_p = N_m d_m + FP + \frac{V}{B}\frac{\mathrm{d}P}{\mathrm{d}t}} \tag{17}$$

The motor torque equation is

$$\boxed{C_m = d_m P = RN_m + J\frac{\mathrm{d}N_m}{\mathrm{d}t}} \tag{18}$$

By eliminating P between these two equations, we get

$$xN_pD_p = N_m d_m + \frac{FR}{d_m}N_m + \frac{FJ}{d_m}\frac{\mathrm{d}N_m}{\mathrm{d}t} + \frac{VR}{Bd_m}\frac{\mathrm{d}N_m}{\mathrm{d}t} + \frac{VJ}{Bd_m}\frac{\mathrm{d}^2N_m}{\mathrm{d}t^2}$$

i.e.

$$\boxed{x\frac{N_pD_p}{d_m} = N_m\left[1+\frac{FR}{d_m^2}\right] + \frac{\mathrm{d}N_m}{\mathrm{d}t}\left[\frac{FJ}{d_m^2}+\frac{VR}{Bd_m^2}\right] + \frac{\mathrm{d}^2N_m}{\mathrm{d}t^2}\left[\frac{VJ}{Bd_m^2}\right]} \qquad (19)$$

or, in operational form (see Chapter 6)

$$\frac{N_m}{x} = \frac{N_pD_p/d_m}{1+\dfrac{FR}{d_m^2}+p\left(\dfrac{FJ}{d_m^2}+\dfrac{VR}{Bd_m^2}\right)+p^2\dfrac{VJ}{Bd_m^2}} \qquad (20)$$

This type of equation occurs frequently in the analysis of hydraulic systems. The equation of the hydraulic servo control is similar in form; a detailed analysis will be found in Chapter 7.

5.6. EQUIVALENT LINEAR EQUATIONS

Replacing a non-linear and non-linearizable equation by an equivalent non-linear equation can only be done under certain conditions, and even then often without strict justification of the validity of such an operation. This will be demonstrated by a simple example.

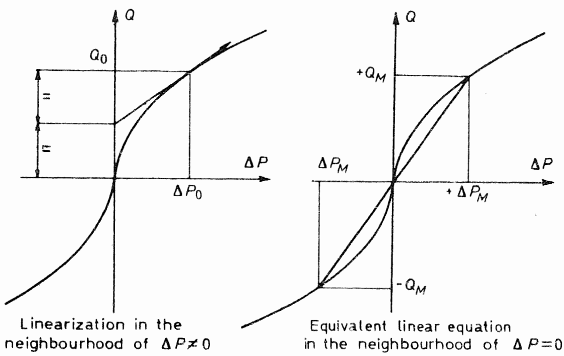


Figure 5.6

5.6.1. EXAMPLE: TURBULENT FLOW IN AN ORIFICE IN THE NEIGHBOURHOOD OF $\Delta P = 0$

Linearization of the equation for turbulent flow through an orifice

$$Q = K \,.\, \sqrt{\Delta P}$$

(or, algebraically, $Q = K\sqrt{|\Delta P|}\,.$sign of ΔP), for points where $\Delta P \neq 0$, can be done by simply taking the tangent to the parabolic characteristic at $\Delta P = \Delta P_0$ (see *Figure 5.6*). But at $\Delta P = 0$, the slope is infinite, and this method of linearization cannot be used.

5.6.1.1. *First approximation to the equivalent linear equation*

However, in a system where ΔP oscillates regularly between $\pm \Delta P_M$, an approximation can be made by replacing the actual physical relationship by the linear relationship obtained by replacing the true curve by the straight line jointing the points $\pm \Delta P_M$ (*Figure 5.6*).

This linear relationship is given by

$$\frac{Q}{Q_M} = \frac{\Delta P}{\Delta P_M}$$

i.e.

$$Q = \frac{Q_M}{\Delta P_M} \Delta P$$

so that

$$Q = \frac{K}{\sqrt{\Delta P_M}} \Delta P \tag{21}$$

Note 1—This relationship is absolutely *arbitrary*: its main justification is its simplicity. The most that can be said for it is that the constant of proportionality, $K/\sqrt{\Delta P_M}$, is the best of those which always underestimate the flow (an over-estimation would be much more difficult). The generalization of this method to other characteristics very different from a parabola may not be justifiable.

Note 2—The very relative validity of eqn. (21) only applies to regular oscillations of ΔP between $\pm \Delta P_M$, but this case is of particular interest, as will be shown in the following Chapter. The most useful dynamic analysis is the study of simple harmonic motion, i.e. of the behaviour of a system which is subjected to sinusoidal variations of fixed amplitude and frequency.

Note 3—By the principle of superposition, the response of a linear system is increased by a factor λ when the input is increased by a factor λ. The *amplitudes* do not appear in the equations and therefore do not affect the analysis. Now, in eqn. (21), given above, the amplitude ΔP_M of ΔP appears in the constant $K/\sqrt{\Delta P_M}$ of the equivalent equation, owing to the fact that the principle of superposition does not apply. Therefore, when a non-linear equation is replaced by an equivalent linear equation, the harmonic analysis must include the parameter of *amplitude* in addition to *frequency*. This will give closer agreement with reality and will enable us to account for phenomena such as limiting oscillation amplitudes and discontinuities in amplitude which would not be predicted by the classical harmonic analysis.

Note 4—The method of equivalent linear equations is not merely a palliative for the analysis of systems which are not amenable to the classical linear methods, but it is also a complementary method giving a more accurate knowledge of real systems for which linearity is limited by the presence of dead zones or saturations.

5.6.1.2. *Better approximations*

For a more accurate harmonic analysis, an equivalent linear equation is required which can represent the actual phenomenon more closely. There are two main ways, one using the describing function approach (approximation of the first harmonic), the other the equivalent energy approximation.

5.6.1.3. *Describing functions*

If $\Delta p = \Delta P_M \sin \omega t$

$$Q = K\sqrt{|\Delta P|}\,.\text{sign of } \Delta P^* = K\sqrt{\Delta P_M}\sqrt{|\sin \omega t|} \times \text{sign of } \Delta P$$

can be expressed as a Fourier series:

$$Q = K\sqrt{\Delta P_M}\,[a_1 \sin \omega t + a_2 \sin 2\omega t + \cdots + a_n \sin n\omega t + \cdots]$$

The describing function approach consists of neglecting all the terms of the series with the exception of the first harmonic, $a_1 \sin \omega t$. This approximation cannot be justified theoretically, although the successive coefficients a decrease with n and, owing to the increasing frequency of the successive terms, their effect will be damped by the action of the various components of the system acting as bypass filters. In the example given

$$a_1 = \frac{4}{\pi}\int_0^{\pi/2} \sin^{3/2} \omega t \, \mathrm{d}(\omega t) = \frac{4}{\pi}\,0{\cdot}8740 = 1{\cdot}113$$

so that

$$Q = 1{\cdot}113\, K\sqrt{\Delta P_M} \sin \omega t,$$

$$Q = 1{\cdot}113 \frac{K}{\sqrt{\Delta P_M}} \Delta P_M \sin \omega t = 1{\cdot}113 \frac{K}{\sqrt{\Delta P_M}} \Delta P$$

It is interesting to compare the coefficients of the describing function with that of the straight-line approximation (Section 5.6.1.1). The ratio is 1·113.

5.6.1.4. *The approximation of equivalent energy*

The mathematical nature of the describing function method should not obscure the fact that it is still arbitrary. In order to give a physical meaning to the equivalent linear equation, the method of equivalent energy is often used.

Suppose that the orifice concerned is present in the piston of a servo-control mechanism. It will be shown in Chapter 7 that this hole has a stabilizing effect. It can be appreciated that the latter is related to the energy dissipated by the liquid as it passes through the orifice. For purposes of calculation, assume that the turbulent flow is replaced by laminar flow through an equivalent orifice

* Often written $Q = K\Delta P/\sqrt{\Delta P}$.

(linear flow as a function of ΔP) which dissipates the same energy as the turbulent flow when ΔP moves through a cycle of amplitude ΔP_M in a time of $2\pi/\omega$.

Thus, if we put $Q = A\,\Delta P$

$$W_1 = 2\int_0^{\pi/\omega} Q\,\Delta P\,dt = 2\,A\,\Delta P_M^2 \int_0^{\pi/\omega} \sin^2 \omega t\,dt = \frac{\pi\,A\,\Delta P_M^2}{\omega}$$

The real flow would have given

$$W_2 = 2\int_0^{\pi/\omega} Q\,\Delta P\,dt = \frac{2\,K\,\Delta P_M^{3/2}}{\omega}\int_0^{\pi} \sin^{3/2} \omega t\,d(\omega t)$$

so that

$$A = \frac{4\,K}{\pi}\,(\Delta P_M)^{1/2}\int_0^{\pi/2} \sin^{3/2} \omega t\,d(\omega t)$$

It can be seen that in this case the equivalent energy approximation and the describing function method give the same value of the coefficient.

Note—To be completely accurate, it should be noted that due simply to the existence of a non-linear component in the servo-control mechanism (in this case, the orifice under consideration), a sinusoidal input does not correspond, as we have supposed, to a strictly sinusoidal ΔP.

Although the presence of the orifice produces a correction factor, ΔP nevertheless approximates very closely to a sinusoidal curve. Some authors suppose that the flow is sinusoidal and use the first harmonic corresponding to ΔP. They put

$$\Delta P = \frac{Q_M^2}{K^2} \sin^2 \omega t \text{ . sign of } Q,$$

which gives

$$\Delta P = \frac{Q_M^2}{K^2}\,\frac{8}{3\pi}\,Q$$

i.e.

$$Q = \frac{3\pi}{8}\,\frac{K}{\sqrt{\Delta P_M}}\,\Delta P \qquad = 1{\cdot}178\,\frac{K}{\sqrt{\Delta P_M}}\,\Delta P$$

The small difference between the result given above and this, derived from an apparently less logical hypothesis, is a confirmation of the practical value of the first calculation.

5.6.2. GENERALIZATION

The methods described hold good for most non-linearizable equations, in particular for all relationships of the form

$$y = K|x|^n \text{.sign of } x \quad (n > 0 \text{ and } \neq 1)$$

in the neighbourhood of $x = 0$, and likewise for physical relationships with dead zones and saturations (cf. *Figure 5.7*) where

$$\begin{array}{lll} y = 0 & \text{for} & -x_0 < x < x_0 \\ y = K(x+x_0) & & -X_0 < x < -x_0 \\ y = K(x-x_0) & & x_0 < x < X_0 \\ y = -K(X_0-x_0) & & x < -X_0 \\ y = +K(X_0-x_0) & & x > X_0 \end{array}$$

The *equivalent linear equation*, $y = K'x$, is valid only for $x = x_M \sin \omega t$.

Figure 5.7

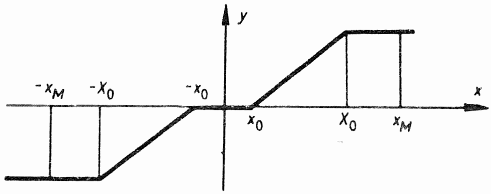


Table 5.1 gives the values of K' (obviously a function of x_M), calculated by the first, crude approximation method of the chord and, in some instances, by the describing function approach.

K', the gain of the equivalent linear equation, *in*creases with x for $y \Rightarrow Kx^n$ with $n > 1$ or for a dead zone; it *de*creases with x for $y = Kx^n$ with $n < 1$ or for a saturation. This is important when considering the stability of a servo system. We will see that instability tends to appear when the gain of an open chain is increased. If several non-linear components are represented by their equivalent equations, it is interesting to estimate the amplitude of the control signal corresponding to the maximum value of the product $K_1' K_2' \ldots$

During an experimental investigation of a system having non-linear components, it is always worth while looking for *regions of instability* by varying the gain, and *measuring the amplitude* of the sustained oscillations, so that the *theoretical and experimental* results for the transfer function at this amplitude can be compared. Any differences between linear theory and experiment can usually be accounted for.

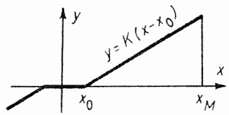
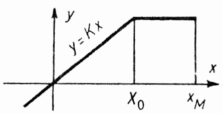
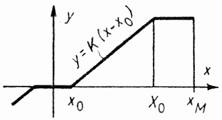
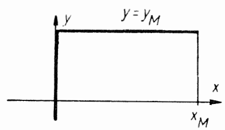
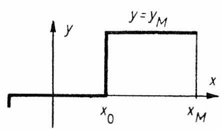
Example—The linear analysis of an electrohydraulic servo control for an aircraft predicted the presence of instability at a frequency of 150 c/s above a certain value of the gain (see Sections 6.6 and 6.7).

The experimental harmonic analysis, made with an input amplitude e, showed a phase difference of 180° at a frequency of about 140 c/s. Increasing the gain, however, resulted in the onset of instability at a frequency of 60 c/s. The oscillation amplitude, e_0, was very much less than e.

The harmonic analysis for $e = e_0$ with a non-linear calculation, taking into account the dead zone of the control potentiometer, confirmed the existence of a phase difference of 180° at 60 c/s. The installation of a better potentiometer increased the oscillation frequency to 110–120 c/s, thus confirming the approach used.

Table 5.1

Functions	*Values of K'* (equivalent equation: $y = K'x$)	
	Approximation of the chord $K' = \dfrac{y_M}{x_M}$	Describing function approach $K' = \dfrac{4}{\pi}\dfrac{1}{x_M}\int_0^{\pi/2} f(x) \sin \omega t \, d(\omega t)$ (for odd functions)
$y = K\|x\|^n \cdot$ sign of x	$K x_M^{n-1}$	$\dfrac{4K}{\pi} x_M^{n-1} \int_0^{\pi/2} \sin^{n+1} \omega t \, d(\omega t)$
$n = 1/3$	$K x_M^{-2/3}$	$\dfrac{4K}{\pi}\, 0{\cdot}9108\, x_M^{-2/3} = 1{\cdot}159\, K x_M^{-2/3}$
$n = 1/2$	$K x_M^{-1/2}$	$\dfrac{4K}{\pi}\, 0{\cdot}8740\, x_M^{-1/2} = 1{\cdot}113\, K x_M^{-1/2}$
$n = 2/3$	$K x_M^{-1/3}$	$\dfrac{4K}{\pi}\, 0{\cdot}8413\, x_M^{-1/3} = 1{\cdot}071\, K x_M^{-1/3}$
$n = 3/2$	$K x_M^{+1/2}$	$\dfrac{4K}{\pi}\, 0{\cdot}7189\, x_M^{+1/2} = 0{\cdot}915\, K x_M^{+1/2}$
$n = 2$	$K x_M$	$\dfrac{4K}{\pi}\,\dfrac{2}{3}\, x_M = 0{\cdot}849\, K x_M$
$n = 3$	$K x_M^{+2}$	$\dfrac{4K}{\pi}\,\dfrac{3\pi}{16}\, x_M^2 = 0{\cdot}750\, K x_M^{+2}$

Linear equations with dead zones and saturations			
Dead zone $\pm x_0$ $(x_M > x_0)$		$K\left(1 - \frac{x_0}{x_M}\right)$	$\frac{2K}{\pi}\left[\frac{\pi}{2} - \sin^{-1}\frac{x_0}{x_M} - \frac{x_0}{x_M}\sqrt{1 - \frac{x_0{}^2}{x^2{}_M}}\right]^*$
Saturation $\pm X_0$ $(x_M > X_0)$		$K\frac{X_0}{x_M}$	$\frac{2K}{\pi}\left[\sin^{-1}\frac{X_0}{x_M} + \frac{X_0}{x_M}\sqrt{1 - \frac{X_0{}^2}{x^2{}_M}}\right]^*$
Dead zone $\pm x_0$ and Saturation $\pm X_0$ $(x_M > X_0)$		$K\frac{(X_0 - x_0)}{x_M}$	$\frac{2K}{\pi}\left\{\sin^{-1}\frac{X_0}{x_M} - \sin^{-1}\frac{x_0}{x_M} + \frac{X_0}{x_M}\sqrt{1 - \frac{X_0{}^2}{x^2{}_M}} - \frac{x_0}{x_M}\sqrt{1 - \frac{x_0{}^2}{x^2{}_M}}\right.^*$
Step input $y = y_M$ sign of x		$\frac{y_M}{x_M}$	$\frac{4}{\pi}\frac{y_M}{x_M}$
Step input with dead zone $\pm x_0$ $(x_M > x_0)$		$\frac{y_M}{x_M}$	$\frac{4}{\pi}\frac{y_M}{x_M}\sqrt{1 - \frac{x_0^2}{x_M^2}}^*$

* See corresponding graphs in Appendix 5.2.

5.7. FORMING AN EMPIRICAL OR SEMI-EMPIRICAL EQUATION

It is usual to make use of at least some existing components in the construction of a complex hydraulic system, but normally certain components have to be tested individually.

In the analysis of the whole system, the individual experimental results are used in the way described below. It will be shown that one of the main advantages of the harmonic method is the parallel use of theoretical analysis and experimental information, both presented in the form of transfer functions. But it is often possible to confirm the experimental results directly from the equations of the component parts, without the use of any theoretical considerations. A complete description of the methods which can be used is beyond the scope of this book, but the following example is characteristic of them.

Consider the design of a fuel regulator for a given jet engine. It is necessary to know the variation of ω with the flow of fuel at each rotational speed ω_0, and for each value of the external parameters (p_0, T_0, etc.), i.e.

$$\Delta\omega = f(\Delta q)$$

A change in the flow of fuel produces a change of engine torque, C_M, but C_M varies with ω at constant fuel flow

$$\Delta C_M = k_1 \Delta q + k_2 \Delta\omega$$

This change in engine torque is balanced by the change in the resisting couple and by the inertia couple

$$\Delta C_M = k_3 \Delta\omega + I \frac{d\Delta\omega}{dt}$$

Eliminating ΔC_M from these two equations gives

$$A \Delta q = \Delta\omega + T \frac{d\Delta\omega}{dt} \tag{22}$$

where

$$\frac{k_1}{k_3 - k_2} = A \qquad \frac{I}{k_3 - k_2} = T$$

Estimation of A and T is complicated and not very accurate, especially at points remote from the operating point. It is preferable that they be found experimentally. A can be found directly by plotting the characteristic $\omega = f(q)$ for successive stable regions of the jet engine during its tests.

Several methods are available for the determination of T. Suppose that, starting in the stable region q_0, ω_0, the flow of fuel is progressively increased, e.g. according to the law $\Delta q = \alpha t$, and the values of Δq and $\Delta\omega$ are recorded at regular intervals of time. The results will be similar to the curves shown in *Figure 5.8*.

Now the integration of eqn. (22) with the initial conditions

$$\Delta\omega = \frac{d\,\Delta\omega}{dt} = 0 \quad \text{for} \quad t = 0$$

gives the relationship

$$\Delta\omega = A\,\alpha\,[t - T\,(1 - e^{-t/T})]$$

The curve of $\Delta\omega$ against t has the asymptote $\Delta\omega = A\alpha(t - T)$ which intersects the t axis at an ordinate $t = T$. This represents a graphical method of evaluating T. It is very simple but not extremely accurate, since it presumes the exact validity of eqn. (22).

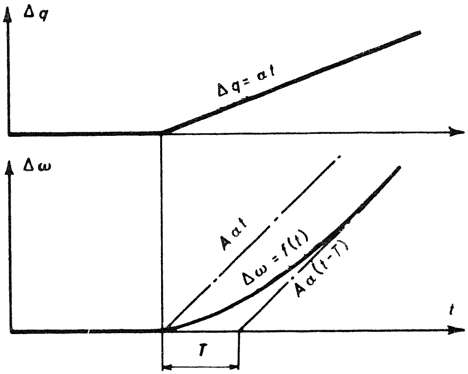


Figure 5.8.

5.8. EQUATIONS OF THE BASIC HYDRAULIC COMPONENTS

Hydraulic systems and devices can be reduced to a certain number of basic components, i.e. components having a single inlet and outlet and defined by the relationship between the *pressure difference* across them and the *flow* through them. The main basic components are listed below, together with their equations and symbolic diagrams.

5.8.1. ACTIVE COMPONENTS

Sources of pressure: accumulators, perfect centrifugal pumps (Section 3.3.3 and *Figure 5.9*)

$$\boxed{P_2 - P_1 = \Delta P = \text{constant}}$$

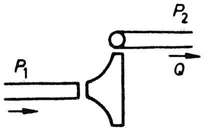


Figure 5.9

Sources of flow: perfect positive displacement pumps (Section 3.3.3 and *Figure 5.10*)

$$\boxed{Q = \text{constant}}$$

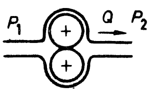


Figure 5.10

5.8.2. NON-ACTIVE COMPONENTS

(*a*) *Laminar orifices* (Section 1.4.1.3 and *Figure 5.11a*)

$$\boxed{Q = Ks\,\Delta P \quad \left(\text{or } \Delta P = \frac{1}{Ks} Q\right)}$$

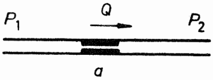


Figure 5.11a

(*b*) *Turbulent orifices* (Section 1.4.1.1 and *Figure 5.11b*)

$$\boxed{Q = Ks\sqrt{\Delta P} \quad \left(\text{or } \Delta P = \frac{Q^2}{K^2 s^2}\right)}$$

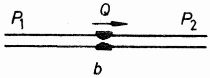


Figure 5.11b

(*c*) *Piston with restoring spring* (without friction, inertia or leakage; incompressible liquid: *Figure 5.12*). If S is the area of the piston, r the restoring stiffness and x the intermediate variable, the displacement of the piston, we have

$$\text{force equation:} \quad rx = S\Delta P$$
$$\text{flow equation:} \quad Q = S\,\mathrm{d}x/\mathrm{d}t$$

Thus, by eliminating x

$$\boxed{Q = \frac{S^2}{r}\frac{\mathrm{d}\Delta P}{\mathrm{d}t} \quad \left(\text{or } \Delta P = \frac{r}{S^2}\int Q\,\mathrm{d}t\right)}$$

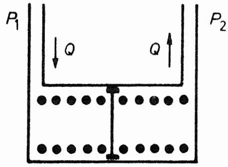


Figure 5.12

(*d*) *Piston with viscous friction* (without inertia, restoring stiffness or leakage; incompressible liquid: *Figure 5.13*). If S is the area of the piston and f the coefficient of friction

$$\text{force equation:} \quad f\,\mathrm{d}x/\mathrm{d}t = S\Delta P$$
$$\text{flow equation:} \quad Q = S\,\mathrm{d}x/\mathrm{d}t$$

Thus, by eliminating x

$$\boxed{Q = \frac{S^2}{f}\Delta P \quad \left(\text{or } \Delta P = \frac{f}{S^2} Q\right)}$$

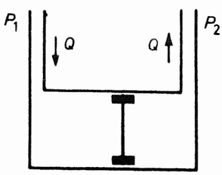


Figure 5.13

(*e*) *Free piston* (with mass but without restoring force, friction or leakage; incompressible liquid: *Figure 5.14*). If S is the area of the piston and M the total mass of the piston unit

$$\begin{aligned} &\text{force equation:} \quad & S\Delta P &= M\,\mathrm{d}^2x/\mathrm{d}t^2 \\ &\text{flow equation:} \quad & Q &= S\,\mathrm{d}x/\mathrm{d}t \end{aligned}$$

Thus, by eliminating x

$$\boxed{\begin{gathered} Q = \frac{S^2}{M}\int \Delta P\,\mathrm{d}t \\ \left(\text{or } \Delta P = \frac{M}{S^2}\,\frac{\mathrm{d}Q}{\mathrm{d}t}\right) \end{gathered}}$$

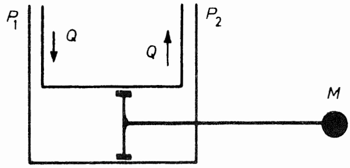


Figure 5.14

(*f*) *Deformable container* (Section 5.3.2, *Figure 5.15*). By definition of the coefficient of elasticity, $\mathrm{d}V = k\,\mathrm{d}P$, we have

$$\boxed{Q = k\,\frac{\mathrm{d}P}{\mathrm{d}t}}$$

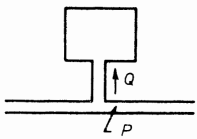


Figure 5.15

(*g*) *Effect of compressibility of the liquid* (Section 5.3.2, *Figure 5.16*)

$$\mathrm{d}V = \frac{V_0}{B}\,\mathrm{d}P$$

so that

$$\boxed{Q = \frac{V_0}{B}\,\frac{\mathrm{d}P}{\mathrm{d}t}}$$

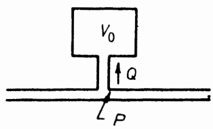


Figure 5.16

It can be seen that the effect of compressibility is equivalent to the effect of elasticity in the walls of the vessel. The liquid can therefore be considered as incompressible, provided that the coefficient of elasticity of the container is increased by $k = V_0/B$.

5.8.3. COMPONENTS CONSIDERED AS A COMBINATION OF BASIC COMPONENTS

The following few examples show how a component can be considered as a combination of the basic components.

Centrifugal pump (*Figure 5.17*)—In its normal operating range, i.e. in the region where ΔP decreases as Q increases

$$\Delta P = P_3 - P_1 = P_2 - P_1 - (P_2 - P_3)$$

$$\boxed{\Delta P = C - Ksf(Q)}$$

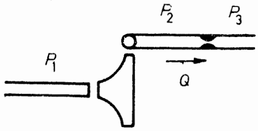


Figure 5.17

where C is a constant.

Positive displacement pump (*Figure 5.18*)—Q decreases with ΔP

$$Q = Q_1 - Q_2$$

$$Q_1 = C$$

$$Q_2 = \frac{1}{Ks} f(\Delta P)$$

$$\boxed{Q = C - \frac{1}{Ks} f(\Delta P)}$$

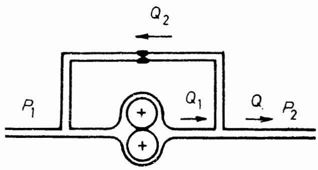


Figure 5.18

where C is a constant.

Piston with restoring spring, friction and inertia (without leakage; incompressible liquid: *Figure 5.19*)

force equation:

$$rx + f\frac{dx}{dt} + M\frac{d^2x}{dt^2} = S\,\Delta P$$

flow equation: $Q = S\,dx/dt$

so that

$$\boxed{\Delta P = \frac{r}{S^2}\int Q\,dt + \frac{f}{S^2}Q + \frac{M}{S^2}\frac{dQ}{dt}}$$

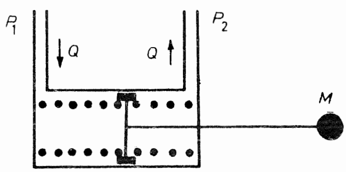
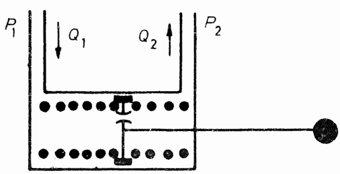


Figure 5.19

This equation is the sum of the inversed equations of (c), (d) and (e).

Piston with restoring spring, friction, inertia, leakage flow and a compressible liquid (*Figure 5.20*).

Figure 5.20



Let V_1 and V_2 be the volumes of the chambers on either side of the piston.

Referring back to the preceding case (incompressible liquid with no leakage flow) and adding a deformable container in both the inlet and outlet piping, and a connecting pipe with an orifice between the inlet and outlet lines, gives the following equations (cf. *Figure 5.21*)

$$Q_1 = Q + Q' + Q_1''$$

$$Q_2 = Q + Q' - Q_2''$$

$$\Delta P = \frac{r}{S^2}\int Q\,\mathrm{d}t + \frac{f}{S^2}Q + \frac{M}{S^2}\frac{\mathrm{d}Q}{\mathrm{d}t}$$

$$Q' = K s f(\Delta P)$$

$$Q_1'' = \frac{V_1}{B}\frac{\mathrm{d}P_1}{\mathrm{d}t}$$

$$Q_2'' = \frac{V_2}{B}\frac{\mathrm{d}P_2}{\mathrm{d}t}$$

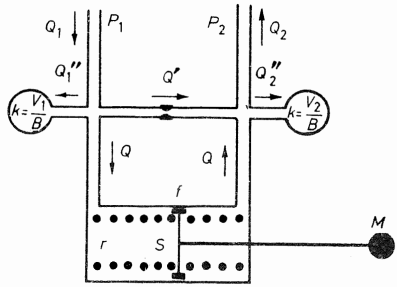


Figure 5.21

APPENDIX 5.1

ELECTRO-HYDRAULIC ANALOGIES

1. THE CONCEPT OF AN ANALOGY

Given a hydraulic circuit and the system of differential equations which represent it mathematically, we may well ask whether there exist one or more circuits, electrical or mechanical for example, which are represented by the same system of differential equations, if, starting with the corresponding relationships between the basic hydraulic components and the electrical or mechanical equivalents, it is possible to construct a circuit having the same mathematical representation as some hydraulic circuit, i.e. to define a hydro-electrical or hydro-mechanical analogy, and further, whether this analogy has any theoretical or practical use.

2. ESTABLISHMENT OF THE $P \to V$, $Q \to I$ ANALOGY

The basic hydraulic components, i.e. those having a single inlet and outlet, are represented by a relationship between the pressure difference, ΔP, between the inlet and outlet, and the flow, Q.

In a similar manner, the basic electrical components are represented by a relationship between the potential difference, ΔV, across the terminals and the current flowing, I.

Complex hydraulic components or hydraulic circuits can be considered as combinations of basic components; the differential equations which represent them are obtained by the application of the laws of *networks* and *junctions*:

law of networks: $\Sigma\Delta P_i = 0$ around a closed circuit

law of junctions: $\Sigma Q_i = 0$ at a junction

The electrical laws of networks and junctions are absolutely analogous:

law of networks: $\Sigma\Delta V_i = 0$ around a closed circuit

law of junctions: $\Sigma I_i = 0$ at a junction

Therefore, with an analogy in which *voltage corresponds to pressure and electric current to flow*, known as the $P \to V$, $Q \to I$ *analogy*, we can construct an electrical circuit analogous to any hydraulic one, by synthesis, starting from a knowledge of the electrical components which are analogous to the basic hydraulic components.

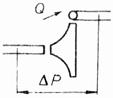
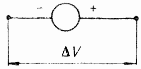
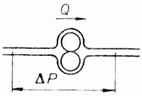
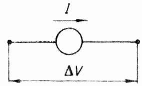
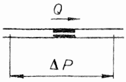
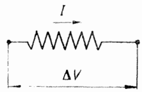
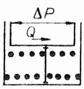
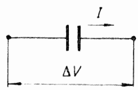
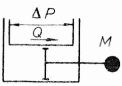
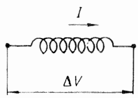
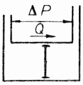
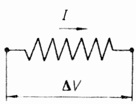
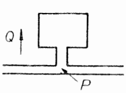
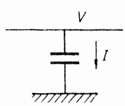
Using the equations which represent the basic hydraulic components, a list has been made showing their correspondence with the equivalent electrical components, using the $P \to V$, $Q \to I$ electro-hydraulic analogy (*Table 5.2*). This Table enables us to construct electrical analogies for a certain number of hydraulic components.

Consider, for example, a piston with a restoring spring of stiffness r, friction f, mass M, and laminar leakage flow Ks, operating with a compressible liquid (bulk modulus, B). The diagram and equations for the piston are given in Section 5.8.3. It is compared with its electrical analogue in *Figure 5.22*.

It should be observed, however, that a certain number of simple hydraulic components do not have *simple* electrical analogies.

The most important of these is the *orifice with turbulent flow* ($Q = Ks\sqrt{\Delta P}$),

Table 5.2

Hydraulic element	*Correspondence*	*Electrical element*
Pressure source  $\Delta P = \text{const.}$	$\Delta P \leftrightarrow \Delta V$	Voltage source  $\Delta V = \text{const.}$
Flow source  $Q = \text{const.}$	$Q \leftrightarrow I$	Current source  $I = \text{const.}$
Laminar orifice  $Q = K s \Delta P$	$K s \leftrightarrow \frac{1}{R}$	Resistance  $I = \frac{1}{R} \Delta V$
Piston with restoring springs  $Q = \frac{S^2}{r} \frac{d\Delta P}{dt}$	$\frac{S^2}{r} \leftrightarrow C$	Capacitance  $I = C \frac{d\Delta V}{dt}$
Free piston  $Q = \frac{S^2}{M} \int \Delta P \, dt$	$\frac{S^2}{M} \leftrightarrow \frac{1}{L}$	Inductance  $I = \frac{1}{L} \int \Delta V \, dt$
Piston with viscous friction  $Q = \frac{S^2}{f} \Delta P$	$\frac{S^2}{f} \leftrightarrow \frac{1}{R}$	Resistance  $I = \frac{1}{R} \Delta V$
Compressibility of liquid or elasticity of container  $Q = \frac{V_0}{B} \frac{dP}{dt}$ $Q = k \frac{dP}{dt}$	$\frac{V_0}{B} \leftrightarrow C$ $k \leftrightarrow C$	Capacitance  $I = C \frac{dV}{dt}$

but it is also true of relief valves (Section 3.2.2.1) and, in general, of components in which hydraulic force is used to move an element in order to vary one or more cross-sectional areas of flow, according to some predetermined relationship.

Finally, the fact that electrical capacitances and inductances are never perfect, sometimes makes the physical establishment of an analogous electrical circuit difficult.

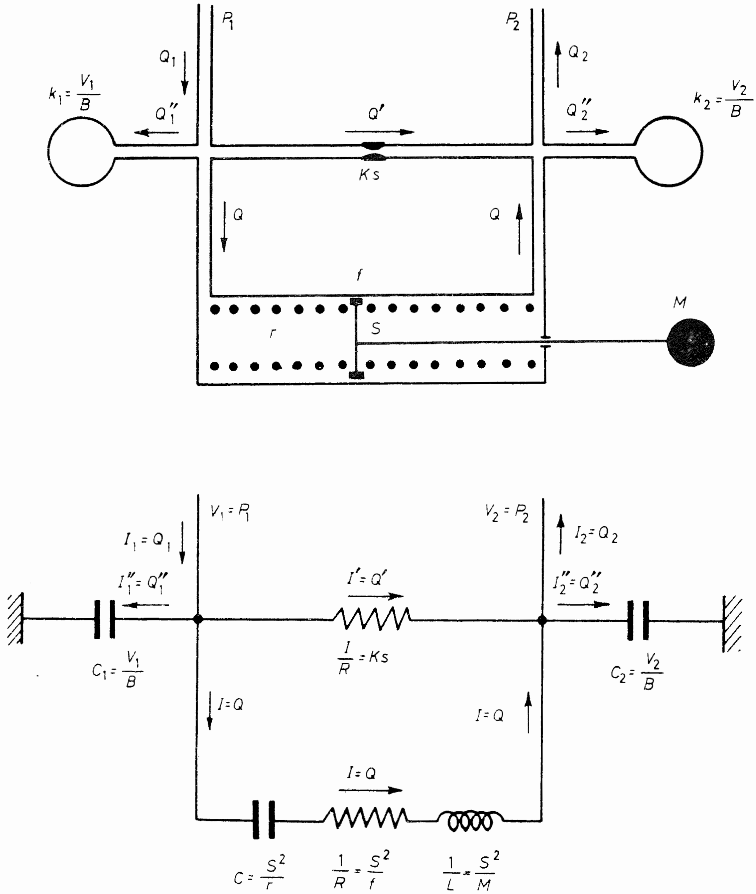


Figure 5.22

3. OTHER ANALOGIES

We may well look for other analogies than pressure → voltage and flow → current. However, it will be seen that any analogies which are not based on the pairs of fundamental quantities (see Section 3.2.2.1), e.g. pressure and flow (whose product is hydraulic power) and voltage and current (whose product is electrical power), are of no use, because fundamental components no longer correspond and because the general concepts, such as those of impedance and power, are not conserved.

There is still the analogy pressure $\rightarrow$ current and flow $\rightarrow$ voltage. This has the disadvantage that the *basic form of the circuits* is not conserved, since there is correspondence between *reciprocal circuits* (networks correspond to junctions and vice versa). Their only advantage is the fact that a piston with inertia only is represented by a pure capacitance, which is easier to achieve in practice than a pure inductance.

Finally, there are two hydro-mechanical analogies, defined by the correspondence between the pairs pressure–flow and velocity–force (whose product is mechanical power):

pressure $\rightarrow$ velocity, flow $\rightarrow$ force, in which the basic form of the circuits is conserved, since the law for networks applies to velocities and the law for junctions applies to forces (projected onto an axis);

pressure $\rightarrow$ force, flow $\rightarrow$ velocity, in which the basic form of the circuits is not conserved but which is apparently attractive owing to the similarity of the fundamental dimensions: pressure and force, and flow and velocity, which differ only by the dimensions of area (L^2).

Tables 5.3 and *5.4* list the factors of correspondence for the four analogies.

The nomenclature for the mechanical systems is: velocity, v, force, F, stiffness, k, coefficient of friction, f, and mass, m.

A difficulty in the hydro-mechanical analogy arises from the fact that the inertia forces in circuits

$$\left[F = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = m\frac{\mathrm{d}v}{\mathrm{d}t}\right]$$

are always relative 'to the ground' (the point of zero velocity).

There is no simple mechanical component which can be represented by the equation $F = m(\mathrm{d}\,\Delta v)/\mathrm{d}t$, so that it is impossible to represent simply e.g. pistons with restoring springs using the $P \rightarrow v$, $Q \rightarrow F$ analogy.

4. THE IMPORTANCE OF THE ELECTRO-HYDRAULIC ANALOGY

The electro-hydraulic analogy is important, since it means that certain hydraulic problems can be solved using an analogue computer. Apart from this particular application, its use is not recommended. It will be noticed that the analogy is not used in this book. However advantageous it may seem to engineers familiar with electrical circuits, the use of the analogy may be dangerous. It obviously leads to an electrical way of thinking and to solving problems in a neat electrical way, which would be ridiculous when translated into their hydraulic equivalent, when there is often a satisfactory hydraulic solution without a simple electrical equivalent.

It is interesting to try to find the reasons for the creative ineffectiveness of the electro-hydraulic analogy, i.e. the fundamental differences between the two systems. Without being too profound, since the problem is really outside the scope of this book, it can be seen that the very material nature of the liquids used in hydraulics is the cause of many of these fundamental differences. There is no hydraulic equivalent of the generator whose function is to start *in motion* the ions which are always *available*: the liquid must be chosen according to its conditions of use—it must be introduced into the circuit, stored, filtered and eventually recuperated, regenerated or replaced after use. Furthermore, owing to the size of hydraulic installations, the liquid has a non-negligible mass which is responsible for the non-linearity of the majority of equations of flow (losses of head proportional to V^2). The scale of pressure also differs from that of voltage, owing to the limit of absolute zero pressure; in any case, the liquid vaporizes before absolute zero is reached. Lastly, the physical properties of

Table 5.3

Analogies	$P \to V$, $Q \to I$ and $P \to v$, $Q \to F$	
Laminar orifice	$Q = K\,S\,\Delta P$	$K\,S \leftrightarrow \frac{1}{R} \leftrightarrow f$
Resistance	$I = \frac{1}{R}\,\Delta V$	
Viscous friction	$F = f\,\Delta v$	
Piston with restoring springs	$Q = \frac{S^2}{r}\,\frac{\mathrm{d}\Delta P}{\mathrm{d}t}$	$\frac{S^2}{r} \leftrightarrow C \leftrightarrow ?$
Capacitance	$I = C\,\frac{\mathrm{d}\Delta V}{\mathrm{d}t}$	
?*		
Free piston	$Q = \frac{S^2}{M}\int \Delta P\,\mathrm{d}t$	$\frac{S^2}{M} \leftrightarrow \frac{1}{L} \leftrightarrow k$
Inductance	$I = \frac{1}{L}\int \Delta V\,\mathrm{d}t$	
Spring	$F = k\int \Delta v\,\mathrm{d}t$	
Piston with viscous friction	$Q = \frac{S^2}{f}\,\Delta P$	$\frac{S^2}{f} \leftrightarrow \frac{1}{R} \leftrightarrow f$
Resistance	$I = \frac{1}{R}\,\Delta V$	
Viscous friction	$F = f\,\Delta v$	
Compressibility of the liquid	$Q = \frac{V_0}{B}\,\frac{\mathrm{d}P}{\mathrm{d}t}$	$\frac{V_0}{B} \leftrightarrow C \leftrightarrow m$
Capacitance relative to earth	$I = C\,\frac{\mathrm{d}V}{\mathrm{d}t}$	
Inertia (necessarily relative to the 'ground')	$F = m\,\frac{\mathrm{d}v}{\mathrm{d}t}$	

* See page 175.

Table 5.4

Analogies $\begin{matrix} P \to I \\ Q \to V \end{matrix}$ and $\begin{matrix} P \to F \\ Q \to v \end{matrix}$		
Laminar orifice	$Q = K\,S\,\Delta P$	
Resistance	$\Delta V = R\,I$	$K\,S \leftrightarrow R \leftrightarrow \dfrac{1}{f}$
Viscous friction	$\Delta v = \dfrac{1}{f}\,F$	
Piston with restoring springs	$Q = \dfrac{S^2}{r}\,\dfrac{\mathrm{d}\Delta P}{\mathrm{d}t}$	
Inductance	$\Delta V = L\,\dfrac{\mathrm{d}I}{\mathrm{d}t}$	$\dfrac{S^2}{r} \leftrightarrow L \leftrightarrow \dfrac{1}{k}$
Spring	$\Delta v = \dfrac{1}{k}\,\dfrac{\mathrm{d}F}{\mathrm{d}t}$	
Free piston	$Q = \dfrac{S^2}{M}\displaystyle\int \Delta P\,\mathrm{d}t$	
Capacitance	$\Delta V = \dfrac{1}{C}\displaystyle\int I\,\mathrm{d}t$	$\dfrac{S^2}{M} \leftrightarrow \dfrac{1}{C} \leftrightarrow ?$
?*		
Piston with viscous friction	$Q = \dfrac{S^2}{f}\,\Delta P$	
Resistance	$\Delta V = R\,I$	$\dfrac{S^2}{f} \leftrightarrow R \leftrightarrow \dfrac{1}{f}$
Viscous friction	$\Delta v = \dfrac{1}{f}\,F$	
Compressibility of the liquid	$Q = \dfrac{V_0}{B}\,\dfrac{\mathrm{d}P}{\mathrm{d}t}$	
Inductance relative to earth	$V = L\,\dfrac{\mathrm{d}I}{\mathrm{d}t}$	$\dfrac{V_0}{B} \leftrightarrow L \leftrightarrow \dfrac{1}{k}$
Spring relative to the ground (fixed end)	$v = \dfrac{1}{k}\,\dfrac{\mathrm{d}F}{\mathrm{d}t}$	

* See page 175.

liquids depend to a large extent on the ambient conditions (viscosity, presence of dissolved air) and on the conditions of use ('wear and tear' of the oil).

Thus industrial hydraulics, being a more 'material' science than industrial electricity, will have less pure laws. The non-linearity of its fundamental equations will retard its development for a long time, especially in respect of its dynamic analyses. It is very far behind electrical technology, and for this reason some authors abuse the analogy.

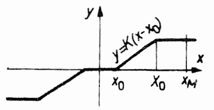
APPENDIX 5.2

DETERMINATION OF THE NUMERICAL COEFFICIENTS OF THE EQUIVALENT LINEAR EQUATIONS

(*Describing function approach*)

Table *5.1* gives equations for the coefficient K' of the equivalent linear equation $y = K'x$. To avoid numerical calculations, the more complicated of these relationships are plotted in the following graphs.

1. *Dead zone*—Curve 1 of *Figure 5.23* gives K'/K as a function of x_M/x_0.



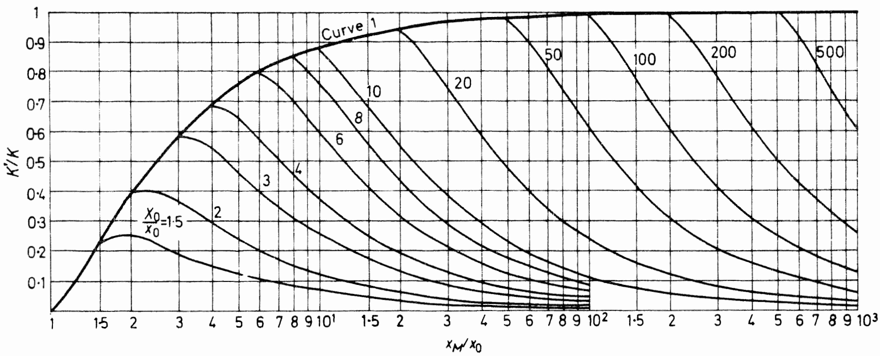


Figure 5.23

2. *Saturation*—The same curve can be used, provided it is this time considered (*Figure 5.24*) as the function

$$1 - \frac{K'}{K} = f\left(\frac{x_M}{x_0}\right)$$

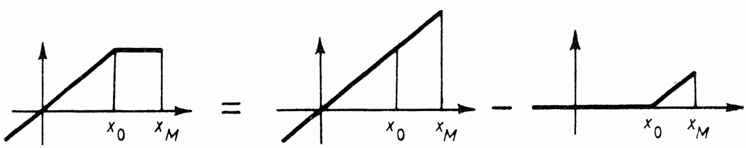


Figure 5.24

3. *Dead zone and saturation*—The other curves in *Figure 5.23* give K'/K as a function of x_M/x_0 for different values of X_0/x_0.

4. *Step input*—K' can be found so easily that a graph is not required.

5. *Step input with a dead zone*—The curve in *Figure 5.25* gives values of $K'/(y_M/x_0)$ as a function of x_M/x_0.

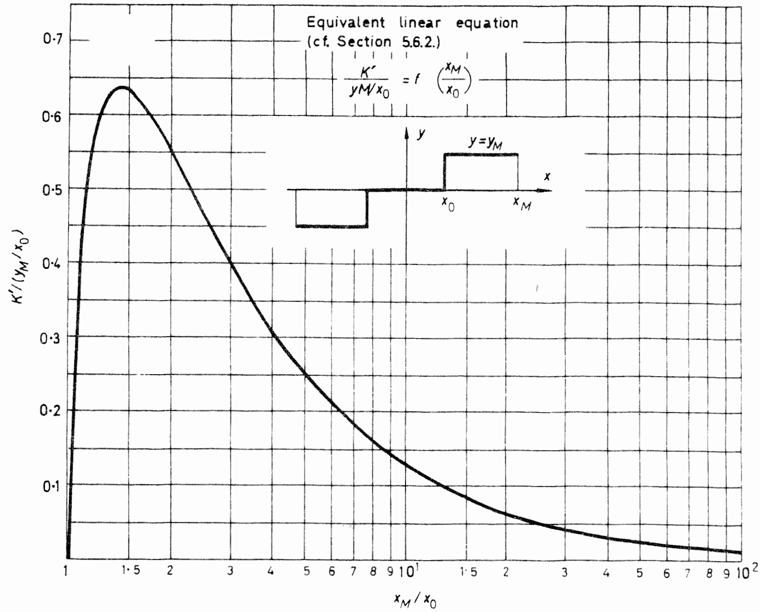


Figure 5.25

6

METHODS OF DYNAMIC ANALYSIS

6.1. THE METHODS OF DYNAMIC ANALYSIS

In general, the formation of the equations for a system produces n integro-differential equations between n unknown functions of time, z_i, and m known functions of time, P_i, together with their successive integrals and derivatives with respect to time.

A knowledge of the system of *integro-differential equations* and *initial conditions* is sufficient for determining the behaviour of the system being studied, i.e. for calculating the n functions $z_i(t)$ corresponding to any system of functions $P_i(t)$.

The classical methods of solution being often impractical, a considerable effort has been made, especially in the last decade or so, to develop *dynamic methods of analysis*, i.e. which either make the solution of the differential equations *easier*, or else methods of extracting from the differential equations important information on the behaviour of the physical system *without actually solving these equations*.

Research has been mainly directed to the analysis of *servo systems*, firstly linear then non-linear. It has resulted in a combination of methods of remarkable efficacy.

The actual development of these methods is beyond the scope of this book. We will simply summarize the fundamental concepts and rules, illustrating them with hydraulic examples. For further examples and developments the reader is referred to more specialized books.

Certain methods which are particularly suited to hydraulic systems will be considered in more detail, essentially those which apply to systems having low damping, since the lack of natural damping is a fairly general characteristic of these systems.

6.2. THE LAPLACE TRANSFORMATION

6.2.1. SIGNIFICANCE

The Laplace transformation is a *mathematical* transformation which substitutes linear algebraic equations for the linear differential equations. With certain exceptions, of which we must at least be aware, its use is sufficiently simple, or perhaps sufficiently mechanical, that no knowledge of the mathematical basis of the transformation is required.

6.2.2. DEFINITION

A function of *time*, t: $x = f(t)$, zero for $t < 0$, is transformed into a function of the *complex variable*, p: $X = F(p)$, which is normally denoted* by

$$X = \mathscr{L} x \tag{1}$$

instead of

$$X(p) = \mathscr{L} x(t)$$

which is accepted as an abbreviated form of

$$X(p) = \int_0^\infty \mathrm{e}^{-pt}\, x(t)\, \mathrm{d}t \tag{2}$$

The Laplace transformation is therefore simply a *change of variable*, from t to p, which is particularly useful owing to certain properties which will be mentioned below.

6.2.3. MAIN PROPERTIES (easily demonstrated from the equation of definition)

1. *Addition*

$$\mathscr{L}(x + y) = \mathscr{L} x + \mathscr{L} y$$

2. *Multiplication by a constant*

$$\mathscr{L} K x = K \mathscr{L} x$$

3. *Differentiation*

$$\mathscr{L} \frac{\mathrm{d}x}{\mathrm{d}t} = p \mathscr{L} x - x_{0+}$$

where x_{0+} is the value of x when the time t is positive and infinitely small.

Important particular case: if $x_{0+} = 0$

$$\mathscr{L} \frac{\mathrm{d}x}{\mathrm{d}t} = p \mathscr{L} x \tag{3}$$

General case:

$$\mathscr{L} \frac{\mathrm{d}^n x}{\mathrm{d}t^n} = p^n \mathscr{L} x - p^{n-1} x_{0+} - p^{n-2} \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{0+} - \cdots - \left(\frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}}\right)_{0+}$$

Important particular case: If $x_{0+} = 0$; $(\mathrm{d}x/\mathrm{d}t)_{0+} = 0$, etc.

$$\mathscr{L} \frac{\mathrm{d}^n x}{\mathrm{d}t^n} = p^n \mathscr{L} x \tag{3'}$$

* It is advisable to use small letters for the physical variables and capital ones for the transformed variables. However, a physical variable and its transformed equivalent are very often given the same symbol, either to avoid the use of unusual symbols or because the risk of confusion is negligible.

A linear differential equation such as

$$y = ax + \beta \frac{dx}{dt} + \gamma \frac{d^2x}{dt^2} + \cdots$$

will be transformed into the algebraic equation

$$\begin{aligned} Y = X(a + \beta p + \gamma p^2 + \cdots) \\ - x_{0+}(\beta + \gamma p + \cdots) \\ - \left(\frac{dx}{dt}\right)_{0+}(\gamma + \cdots) \end{aligned}$$

which if $x_{0+} = 0$; $(dx/dt)_{0+} = 0$, etc. reduces to:

$$Y = X(a + \beta p + \gamma p^2 + \cdots)$$

4. *Integration*

$$\mathscr{L} \int_0^t x \, dt = \frac{1}{p} \mathscr{L} x \tag{4}$$

If the initial conditions result in a constant of integration which is not zero,

$$\int x \, dt = c + \int_0^t x \, dt$$

$$\mathscr{L} \int x \, dt = \frac{c}{p} + \frac{1}{p} \mathscr{L} x$$

5. *Time delay*

Suppose a function $x(t)$ (zero for $t < 0$) is transformed to $\mathscr{L}x(t) = X(p)$. Then the function $x(t - \tau)$ (zero for $t < \tau$) becomes

$$\mathscr{L} x(t - \tau) = e^{-\tau p} X(p)$$

6. *Final value*

Suppose a function $x(t)$ is transformed to $X(p)$ such that $pX(p)$ is an analytic function with no singularities in the right-hand side of the complex plane, including the imaginary axis.

When t tends to infinity, $x(t)$ tends to the limit of $pX(p)$ as p tends to zero

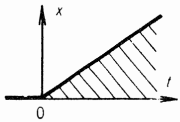
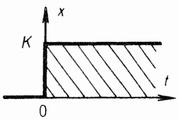
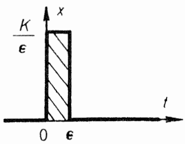
$$\underset{t \to +\infty}{\text{limit of}}\ x(t) = \underset{p \to 0}{\text{limit of}}\ pX(p) \tag{5}$$

This relationship is extremely valuable, since it allows us to determine the final state of a system from a knowledge of the Laplace transform of its variable characteristic, without having to make an inverse Laplace transformation.

7. *Initial value*

$$\underset{t \to 0+}{\text{limit of}}\ x(t) = \underset{p \to \infty}{\text{limit of}}\ pX(p) \tag{6}$$

Table 6.1

$x(t)$			$X(p)$
Impulse		$x = 0$ for $t < 0$ $x = K/\varepsilon$ for $0 < t < \varepsilon$ $x = 0$ for $t > \varepsilon$ and $\varepsilon \to 0$	K
Step		$x = 0$ for $t < 0$ $x = K$ for $t > 0$	$\dfrac{K}{p}$
Velocity step		$x = 0$ for $t < 0$ $x = Kt$ for $t > 0$	$\dfrac{K}{p^2}$
t^n			$\dfrac{n!}{p^{n+1}}$
$e^{-\alpha t}$			$\dfrac{1}{p + \alpha}$
$t\,e^{-\alpha t}$			$\dfrac{1}{(p + \alpha)^2}$
$\sin \omega t$			$\dfrac{\omega}{p^2 + \omega^2}$
$\cos \omega t$			$\dfrac{p}{p^2 + \omega^2}$
$e^{-\alpha t} \sin \omega t$			$\dfrac{\omega}{(p + \alpha)^2 + \omega^2}$
$e^{-\alpha t} \cos \omega t$			$\dfrac{p + \alpha}{(p + \alpha)^2 + \omega^2}$

6.3. TRANSFER FUNCTION AND BLOCK DIAGRAM OF A BASIC COMPONENT

6.3.1. DEFINITION

Consider, in a general way, a component A which establishes a relationship between x and y so that, when an *input* $x = x_i(t)$ is applied, the *output* is $y = y_i(t)$.

The transfer function of the component A for the input $x = x_i(t)$ is the *function of p* given by

$$H_i(p) = \frac{Y_i(p)}{X_i(p)} \tag{7}$$

i.e. the ratio of the Laplace transformed output to the transformed input. Component A and the relationship it establishes between x_i and y_i are conventionally represented by the basic block diagram

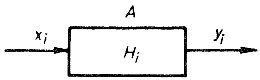


Figure 6.1

Example—For a given car on a straight flat road, there is for each type of accelerator movement $x = x_i(t)$ a corresponding speed behaviour $v = v_i(t)$. The transfer function between x_i and v_i of the given car under the given conditions is

$$H_i = \frac{V_i(p)}{X_i(p)}$$

N.B. Many components, in particular those which rely on an external power source, are *irreversible*, i.e. they can convert x to y but not y to x.

For instance, a simple valve on a constant-pressure reservoir provides a flow which is a function of the valve position, but it is impossible to adjust the position of the valve by the action of the flow. This irreversibility is not apparent from the differential equations.

Even reversible components are normally used only in one predetermined direction, y being controlled by x.

This irreversibility and direction of use are denoted by the expressions *input* and *output* and by the corresponding arrows on the block diagram.

6.3.2. USE OF TRANSFER FUNCTIONS FOR LINEAR COMPONENTS

The usefulness of the transfer function is not apparent from the definition given. There is no reason why a component should not have a different transfer function for each input function $x_i(t)$; this would render the concept of a transfer function trivial. But if the component is linear, i.e. if its operation is represented by a linear differential equation between x and y

$$ax + bx' + \ldots + mx^{(l)} = \alpha y + \beta y' + \ldots + \mu y^{(\lambda)}$$

the relationship between the Laplace transformed functions of x and y is

$$X(a + bp + \cdots + mp^l) - x_{0+}(b + cp + \cdots + mp^{l-1}) - x'_{0+}(c + \cdots + mp^{l-2}) - \cdots - \overset{(l-1)}{x_{0+}}\, m$$
$$= Y(\alpha + \beta p + \cdots + \mu p^\lambda) - y_{0+}(\beta + \gamma p + \cdots + \mu p^{\lambda-1}) - y'_{0+}(\gamma + \cdots + \mu p^{\lambda-2}) - \cdots - \overset{(\lambda-1)}{y_{0+}}\, \mu,$$

which can be written

$$X(a + bp + \cdots + mp^l) - F(p, x_{0+}, x'_{0+}, \ldots, \overset{(l-1)}{x_{0+}}) = Y(\alpha + \beta p + \cdots + \mu p^\lambda) - G(p, y_{0+}, y'_{0+}, \ldots, \overset{(\lambda-1)}{y_{0+}})$$

giving

$$H(p) = \frac{Y}{X} = \frac{a + bp + \cdots + mp^l}{\alpha + \beta p + \cdots + \mu p^\lambda} - \frac{1}{X}\,\frac{(F - G)}{(\alpha + \beta p + \cdots + \mu p^\lambda)} \tag{8}$$

It can be shown that if the component is initially at rest, i.e. if the values of x and y and their successive derivatives are zero at time 0^- (without necessarily being zero at time 0^+)

$$F - G \equiv 0$$

so that

$$\boxed{H(p) = \frac{Y}{X} = \frac{a + bp + \cdots + mp^l}{\alpha + \beta p + \cdots + \mu p^\lambda}} \tag{9}$$

i.e. $H(p)$ is *independent of* X.

A linear component has a unique transfer function, independent of the input for all operations starting from rest.

This property justifies the systematic change of variables recommended in Section 5.1.2

$$\Delta A = A - A_0$$

which has the effect of bringing all the variables to rest at time $t = 0^-$. It is extremely valuable, since it enables us to calculate the response of component A to any given input.

Some important special cases

The Laplace transform of the response of a component of transfer function $H(p)$ to a unit impulse is exactly $H(p)$.

The Laplace transform of the response to a unit step input (see *Table 6.1*) is $H(p)/p$.

The final condition of a component of transfer function $H(p)$ subjected to a unit step input is determined by finding the limit of $H(p)$ as p tends to zero.

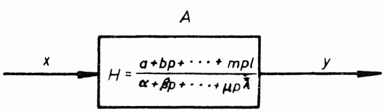


Figure 6.2

The transfer function of a linear component completely characterizes the component; it is often written inside the rectangle representing the component on the block diagram (*Figure 6.2*).

It must be remembered, however, that it is not valid for two cases: (*a*) *when the system is not initially at rest* (as mentioned above), and (*b*) *for inputs which are either too large or too small.*

In fact, no physical system is perfectly linear. As the input amplitude is increased, a state is eventually reached where the output does no longer increase proportionally. The principle of superposition does not hold, and *saturation* is said to take place. The component is no longer linear and the transfer function is no longer valid.

When the input amplitude is decreased, a state is eventually reached where the output becomes zero for finite values of the input. The component is said to have a *dead zone* or *threshold*. The transfer function is zero in the dead zone, and its value differs considerably from normal for inputs only slightly greater than the dead zone.

6.3.3. NON-LINEAR COMPONENTS

For non-linear components with equations which are linearizable near an operating point (cf. Section 5.4), we obviously use the linearized equations to produce a transfer function, which is valid for all values of the input in the neighbourhood of this operating point.

It must be remembered that the linearized equations become more approximate as the point considered moves away from the point of static operation and that the transfer function will become less valid as the amplitude of the input is increased.

For non-linearizable components, the approximation of equivalent linear equations (cf. Section 5.6) also allows us to establish a transfer function. This is (*a*) *never more than an approximation* whose accuracy is difficult to assess, although experience has shown that it is useful in practice; (*b*) it is *valid only for sinusoidal inputs* or for those whose form is similar to a sine curve (cf. Section 5.6), but it is always possible to break down a periodic function into a Fourier series; (*c*) it is a *function of the amplitude of the sinusoidal input considered.*

6.3.4. SUMMARY

Summing up, the transfer function of any component defines a relationship between x and y and is a multiplying factor for obtaining the Laplace transform of y from that of x.

If the operation of the component can be represented with any degree of accuracy and generality by a linear differential equation in x and y, the transfer function deduced from this equation will be valid in the region for which the equation is valid for every variation of the input relative to its initial value.

Linear components will have a unique transfer function in the region between the dead zone and saturation.

Linearizable components will have a different transfer function for each point of linearization.

Non-linearizable components will have a different transfer function for each static operating point and for each input amplitude. The validity of the transfer function is not rigourously established and may not extend to non-sinusoidal inputs.

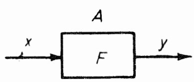
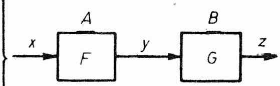
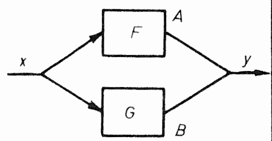
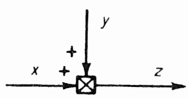
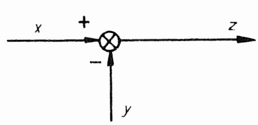
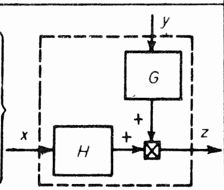
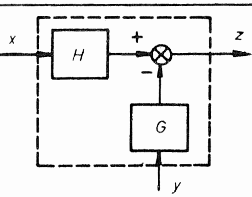
6.4. EXTENSION TO COMPLEX PHYSICAL SYSTEMS

6.4.1. DRAWING BLOCK DIAGRAMS

The block diagram of a physical system consisting of a number of components is a conventional graphic representation of the system, made up of basic diagrams. It is equivalent to the table of differential equations but is generally more informative, especially in the case of a large number of differential equations each containing a small number of variables of parameters; this usually applies to control and servo loops.

The conventions used in the diagrams vary according to the author; those used in this book are shown in *Table 6.2.*

Table 6.2

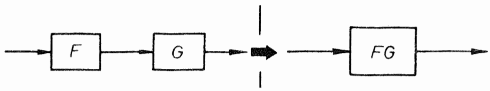
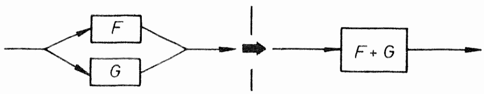
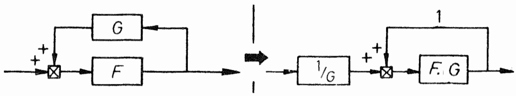
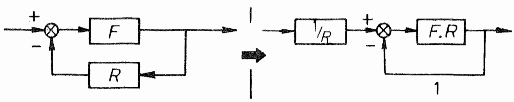
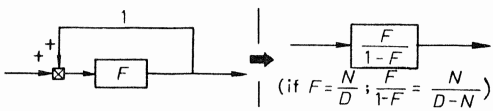
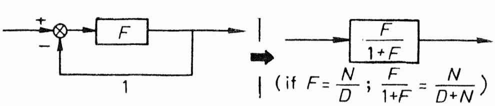
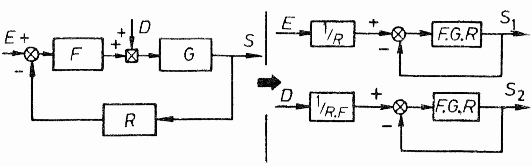
Component A of TF : $\frac{Y}{X}=F$		Two components in series	
		Two components in parallel	
Component giving the sum of two quantities $z=x+y$		Components giving the difference of two quantities $z=x-y$	
Component transforming and summing two quantities $z=h(x)+g(y)$		Component transforming and subtracting two quantities $z=h(x)-g(y)$	

6.4.2. MANIPULATION OF TRANSFER FUNCTIONS AND BLOCK DIAGRAMS

Since transfer functions are multiplying factors, ($Y = H \times X$), the ordinary rules of algebra can be applied to them. In particular, the block diagrams can be reduced to canonical forms with which we will be familiar.

The principal rules of manipulation, which can be easily established, are summarized in *Table 6.3.*

Table 6.3

Equivalent diagrams	*Operation*
	Multiplication
	Addition
	*Reduction to canonical form with unit feedback:* addition
	*Reduction to canonical form with unit feedback:* subtraction
	*Reduction of canonical form with unit feedback:* addition
	*Reduction of canonical form with unit feedback:* subtraction
	Introduction of secondary parameters (perturbations): $S = S_1 + S_2$

6.4.3. SERVO SYSTEMS

As mentioned previously, there is no basic difference between servo systems and other systems.

A servo system is a control system in which the action is continuously controlled by the effective magnitude of the output.

A component called the *adder* measures the difference between the input E and the output S:

$$\varepsilon = E - S$$

or, more generally,

$$\varepsilon = FE - RS$$

where FE and RS are functions of E and S, respectively, and are two quantities of the same physical nature and therefore comparable. These functions are eventually subjected to correction terms.

The difference ϵ is known as the *error* which acts on the next component of the chain.

Thus we can draw the block diagram of a servo system (*Figure 6.3*) and also its canonical form (*Figure 6.4*).

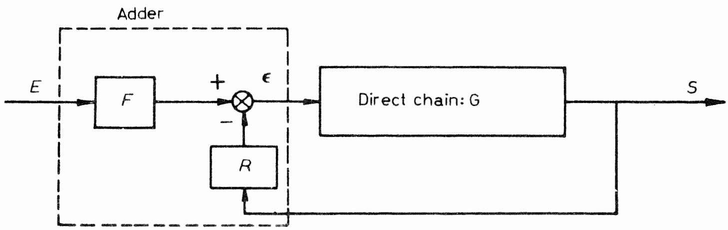


Figure 6.3

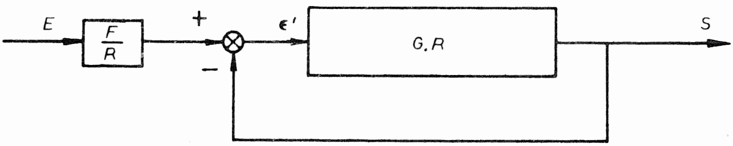


Figure 6.4

The *open-loop* transfer function of a servo system is

$$\frac{S}{\varepsilon'} = GR$$

and the *closed-loop* transfer function

$$\frac{S}{E} = \frac{FG}{1 + GR}$$

6.4.4. EXAMPLE: FUEL SUPPLY SYSTEM OF AN ENGINE

We shall consider a control chain without, and then with, a feedback loop. In the first case, the fuel is supplied directly to the engine, in the second, via a tachometric regulator.

In Section 5.7 it was established that the differential equation relating the angular velocity, ω, of the engine to the flow of fuel, q, is

$$A\,\Delta q = \Delta\omega + T\,\mathrm{d}\Delta\omega/\mathrm{d}t$$

Applying the Laplace transformation, this becomes

$$A\,\Delta Q = \Delta\Omega + Tp\,\Delta\Omega,$$

$$\Delta\Omega = \Delta Q\,\frac{A}{1+Tp}$$

i.e. the diagram of *Figure 6.5*.

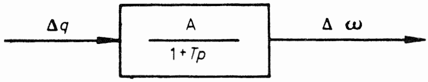


Figure 6.5

Suppose that the engine is mounted on a test bed and supplied from an independent pump at constant pressure, the flow of fuel being controlled by a valve which in turn is remotely controlled from the test console. Let

e = position of fuel control lever on console
y = position of valve opening, $y = f(e)$
q = flow of fuel, $q = g(y)$
F = transfer function of the remote control
G = the transfer function of the valve, and putting $H = A/(1+Tp)$

the block diagram is that of *Figure 6.6*.

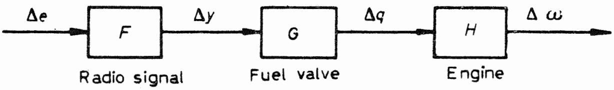


Figure 6.6

Now if the pressure, p_1, of the supply pump is varied, the expression for the flow becomes

$$q = g(y) + \gamma(p_1)$$

and the diagram is that of *Figure 6.7*.

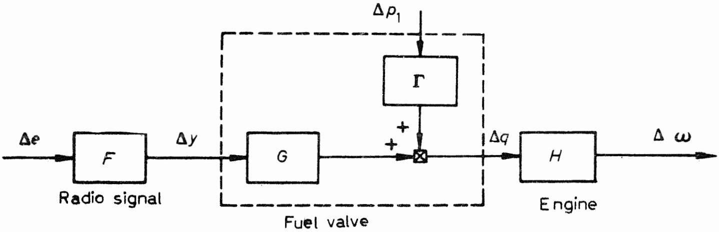


Figure 6.7

If the pump is driven by the engine and its delivery pressure is a function of the engine speed

$$p_1 = l(\omega)$$

the diagram becomes that of *Figure 6.8.*

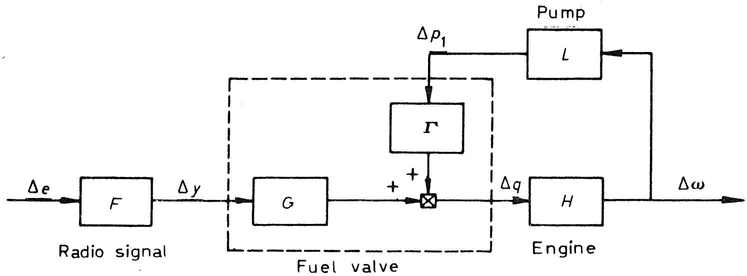


Figure 6.8

Finally, if the engine is equipped with a regulator, such that the position y of the valve is the result of the equilibrium between the force from the centrifugal weights (dependent on ω) and that exerted by a spring connected to the remote control (position x), then the arrangement becomes a *servo system*.

The equation for the tachometric detector (cf. Section 5.4.2.4) is

$$\Delta y = \alpha\,\Delta x - \beta\,\Delta\omega\,,$$

and the block diagram is that of *Figure 6.9.*

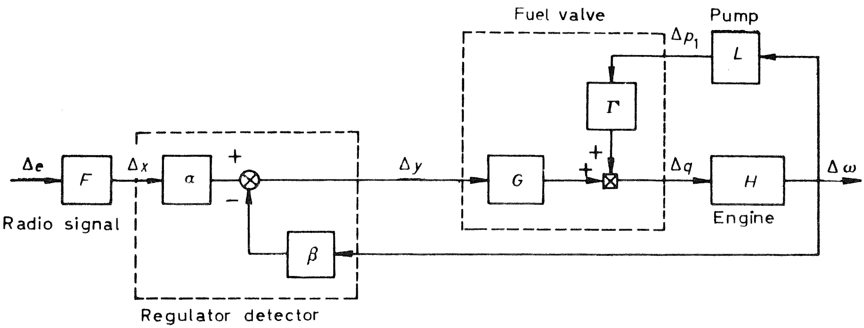


Figure 6.9

Several compensating and amplifying components are usually placed between the detector and the valve. If the collective transfer function of these components is J, the diagram becomes that of *Figure 6.10* or, more conventionally, of *Figures 6.11* and *6.12*.

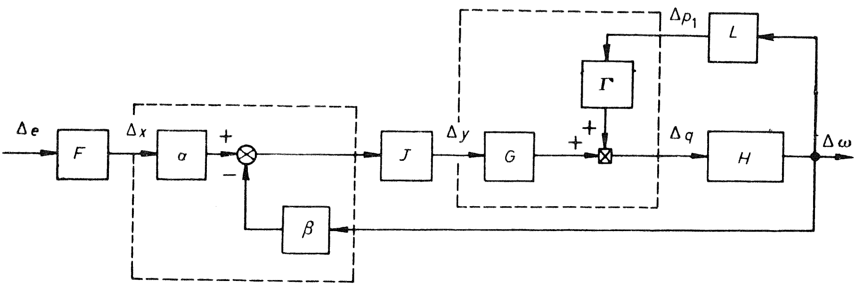


Figure 6.10

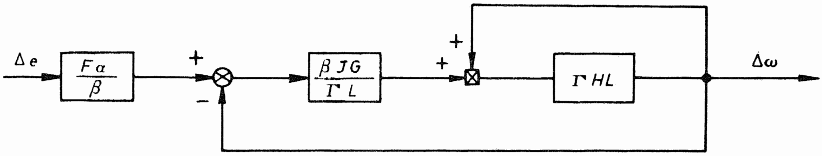


Figure 6.11

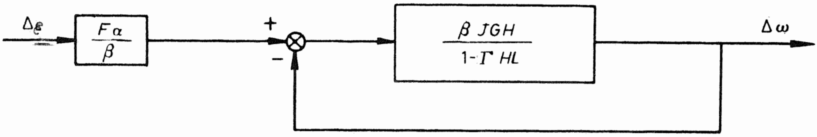


Figure 6.12

6.4.5. EXTENSION OF THE CONCEPT OF TRANSFER FUNCTION. TRANSFER MATRIX

Up to now, we have considered basic components with a single input and output, represented by a single transfer function. But a basic component often has several equations between several inputs and several outputs, in particular, two equations between two inputs and two outputs whose product constitutes the input *power* and output *power*, respectively, i.e. between input and output

voltage and *current* for an electrical component
velocity and *force* for a mechanical component
pressure and *flow* for a hydraulic component.

If these equations are linear, they can be written in the form of the Laplace transformed inputs and outputs, E_1, E_2, S_1 and S_2:

$$S_1 = A\,E_1 + B\,E_2$$
$$S_2 = C\,E_1 + D\,E_2$$

where A, B, C and D are functions of p. This can be written in matrix form

$$\begin{vmatrix} S_1 \\ S_2 \end{vmatrix} = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \times \begin{vmatrix} E_1 \\ E_2 \end{vmatrix} \tag{10}$$

The basic component with two inputs and two outputs is known as a *quadripole* (block diagram *Figure 6.13*).

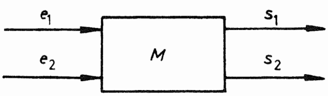


Figure 6.13

The rules of matrix calculation can be applied directly to the transfer matrix:

$$M = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix}$$

The proof of this transcends the scope of this book. The reader who is not familiar with matrix calculation need not be concerned. Its use is necessary only for systems with a degree of complexity which is rarely met with in actual hydraulic problems.

A knowledge of the upstream and downstream characteristics usually eliminates one variable input and output, sometimes at the cost of introducing a slight approximation, so that the component may be treated as a basic component with a single input and output and therefore with a simple transfer function.

EXAMPLE: REGULATOR FOR AN ENGINE WITH VARIABLE EXHAUST NOZZLE

Double regulation is often used for engines with variable nozzles.

A tachometric regulator *detects* the speed of rotation, n, and *adjusts* the flow of fuel, q. It has the transfer function

$$H_N = \frac{\Delta Q}{\Delta N_0 - \Delta N} = \frac{\Delta Q}{\varepsilon_N}$$

A temperature regulator *detects* the temperature at the outlet from the turbine, t_4, and *adjusts* the cross-sectional area of the exhaust nozzle, s. It has the transfer function

$$H_T = \frac{\Delta S}{\Delta T_{40} - \Delta T_4} = \frac{\Delta S}{\varepsilon_T}$$

The engine is therefore a thermodynamic quadripole defined by the equations

$$\Delta N = A\,\Delta Q + B\,\Delta S$$
$$\Delta T_4 = C\,\Delta Q + D\,\Delta S$$

and has the block diagram shown in *Figure 6.14*.

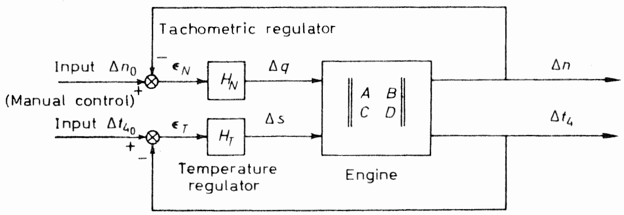


Figure 6.14

We shall examine the tachometric regulator *in the presence of the temperature regulator*. Without the latter, the *open-loop* transfer function of the tachometric regulator (omitting the symbol Δ to make the equation less cumbersome) is

$$\frac{N}{\varepsilon_N} = H_N\,A$$

In the presence of the temperature regulator, we have

$$N = \varepsilon_N H_N A + \varepsilon_T H_T B$$

and

$$T_4 = \varepsilon_N H_N C + \varepsilon_T H_T D$$

If there is no signal to the temperature regulator, i.e. if $\epsilon_T = -T_4$,

$$N = \varepsilon_N H_N A - T_4 H_T B$$

and

$$T_4 (1 + H_T D) = \varepsilon_N H_N C$$

so that, eliminating T_4,

$$\frac{N}{\varepsilon_N} = H_N A \left[1 - \frac{BC}{AD} \frac{H_T D}{1 + H_T D}\right]$$

The effect of the temperature regulator on the tachometric regulator is to multiply its open-loop transfer function by the factor

$$1 - \frac{BC}{AD} \frac{H_T D}{1 + H_T D}$$

In a similar way, it can be shown that the effect of the tachometric regulator on the temperature regulator is represented by the factor

$$1 - \frac{BC}{AD} \frac{H_N A}{1 + H_N A}$$

This analysis can be made without the use of matrices. Although these results have been established for a particular example, they can be applied generally.

6.5. TRANSFER LOCUS

6.5.1. DEFINITION AND USE

Let $H(p)$ be the transfer function of a component or a physical system.

The curve of the complex function $H(j\omega)$, *graduated in* ω, is known as the *transfer locus*; it can be considered as the representation of the imaginary axis in the transformation $p \to H(p)$.

The use of the transfer locus is the basis of the dynamic analysis of linear systems.

Like the transfer function, the transfer locus completely defines the behaviour of a system. It can be found easily: by calculation from the equations, by experiment or by a combination of these, each process being applied to different basic components.

To find the transfer locus of a linear system experimentally merely requires to apply to it inputs

$$e = e_i \sin \omega_i t$$

for a series of values of w and to record the corresponding outputs which under steady conditions have the form

$$s = s_i \sin (\omega_i t + \Phi)$$

The *point corresponding to* ω_i *on the transfer locus has a modulus of* s_i/e_i *and an argument of* ϕ*.

---

* $$H(p) = \frac{S}{E} = \frac{\mathscr{L}[s_i \sin(\omega_i t + \Phi)]}{\mathscr{L}[e_i \sin \omega_i t]} = \frac{\mathscr{L}[s_i \sin \omega_i t \cos \Phi] + \mathscr{L}[s_i \cos \omega_i t \sin \Phi]}{\mathscr{L}[e_i \sin \omega_i t]}$$

$$= \frac{s_i \cos \Phi \,\mathscr{L} \sin \omega_i t + s_i \sin \Phi \,\mathscr{L} \cos \omega_i t}{e_i \mathscr{L} \sin \omega_i t} = \frac{s_i}{e_i} \cos \Phi + \frac{s_i}{e_i} \frac{\mathscr{L} \cos \omega_i t}{\mathscr{L} \sin \omega_i t} \sin \Phi$$

Using the results of *Table 6.1*, this becomes

$$H(p) = \frac{s_i}{e_i}\left[\cos \Phi + \frac{p}{\omega} \sin \Phi\right] \quad \text{et} \quad H(j\omega) = \frac{s_i}{e_i}[\cos \Phi + j \sin \Phi]$$

6.5.2. PRACTICAL REPRESENTATION

Let us express the complex number $H(j\omega)$ in the form $Ae^{j\phi}$ where A and ϕ are functions of ω. There are a number of different ways of plotting this complex number $Ae^{j\phi}$ as a function of ω. The three most commonly used ones are

The Nyquist diagram

$$\text{Classical coordinates}\ \left\{\begin{array}{l}\text{abscissa: } A\cos\phi \\ \\ \text{ordinates: } A\sin\phi\end{array}\right\}\ \text{graduated in } \omega$$

The Nichols plot

$$\begin{array}{c}\text{Semi-logarithmic} \\ \text{coordinates}\end{array}\ \left\{\begin{array}{l}\text{abscissa: } \phi \\ \\ \text{ordinates: } 20\log_{10}A\end{array}\right\}\ \text{graduated in } \omega$$

Frequency response curves

These are curves of amplitude ($20\log_{10}A$) and phase angle (ϕ), both presented as a function of $\log\omega$.

The relative advantages and disadvantages of these three methods of presentation are summarized in *Table 6.4*.

Table 6.4

Operation and use	*Nyquist*	*Nichols*	*Frequency response*
Multiplication by a constant (fundamental problem of gain adjustment)	not easy to use (radial displacement from centre O)	easy: translation parallel to the ordinates axis	easy: translation parallel to the ordinates axis of the amplitude curve
Multiplication of two transfer functions	not practical	easy: addition of abscissae and of ordinates	easy: addition of ordinates of the two curves
Addition of two transfer functions	easy: vectorial addition	fairly complicated (see below)	
Derivation of the closed-loop transfer function with unit feedback from the open-loop transfer function, H: $\dfrac{H}{1+H}$	easy on graphs with Hall's circles (cf. *Figure 6.17*)	easy on graphs with Nichols curves (cf. *Figure 6.16*)	only by using approximate methods
Plotting the curves at infinity and near the origin necessary for the analysis of the stability of closed loops	easy, by application of the usual rules	more difficult than than Nyquist	difficult

Frequency response curves are favoured by electrical engineers. In fact, for transfer functions which can be reduced to a product of binomials

$$H = \frac{(1 + \tau_a p)\,(1 + \tau_b p)\,...}{(1 + \tau_\alpha p)\,(1 + \tau_\beta p)\,...}$$

it is possible to determine two asymptotes for the amplitude curve (*Figure 6.15*):

the horizontal line $20 \log_{10} A = 0$ for $\omega < 1/\tau_i$
a straight line of slope ± 6 dB per octave for $\omega > 1/\tau_i$.

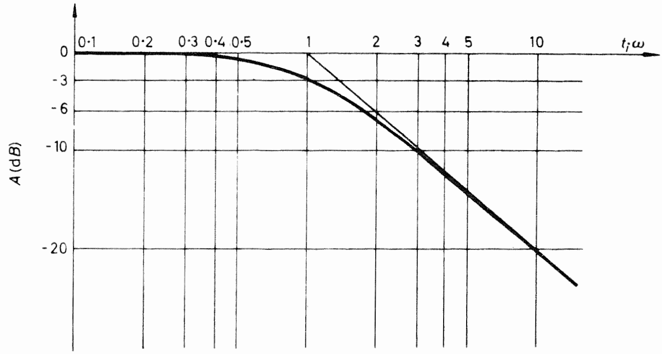


Figure 6.15

The transfer locus is therefore a succession of straight lines of slope $\pm 6K$ dB per octave where K is an integer. This method of presentation is *not suitable for hydraulic problems* where the transfer function very often contains poorly damped trinomials which deviate from the asymptotes:

$$1 + \varepsilon \frac{p}{\omega_i} + \frac{p^2}{\omega_i^2}$$

It should be used only for the *presentation of results* (frequency response curves of a complete hydraulic assembly).

The *Nyquist diagram* can be used in hydraulics to assess the rare problems in which the form of the curve at infinity is not evident.

The *Nichols plot* is the ideal method of presentation. The only difficult operation is the addition of two transfer functions, but this can be facilitated by the use of Nichols curves.

The relationship $G = H+F$ can be written in the form

$$G = H \frac{1}{\dfrac{H}{F} \Big/ \left(1 + \dfrac{H}{F}\right)}$$

so that the addition can be performed by carrying out the following sequence of operations.

division:

$$H_1 = \frac{H}{F}$$

finding the closed-loop function from the open-loop value by reading directly off Nichols curves

$$H_2 = \frac{H_1}{1 + H_1}$$

division:

$$G = \frac{H_1}{H_2}$$

Figures 6.16–6.18 show the open-loop transfer locus of an electrohydraulic servo control presented in these three different ways for two values of gain ($f = \omega/2\pi$, in hertz).

The closed-loop transfer function is found from the Nyquist and Nichols plots by reading off the value directly, using the curvilinear coordinates formed by Hall circles and Nichols curves, respectively (cf. Chapter 11, Graphs *I* and *J*).

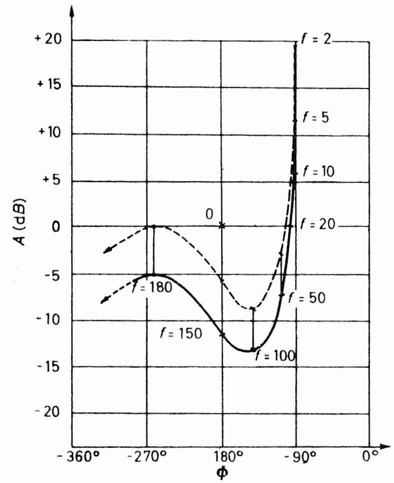


Figure 6.16. Nichols plot

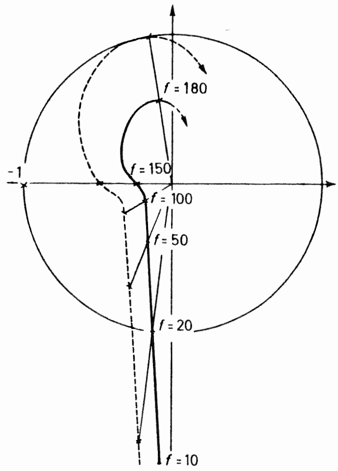


Figure 6.17. Nyquist plot

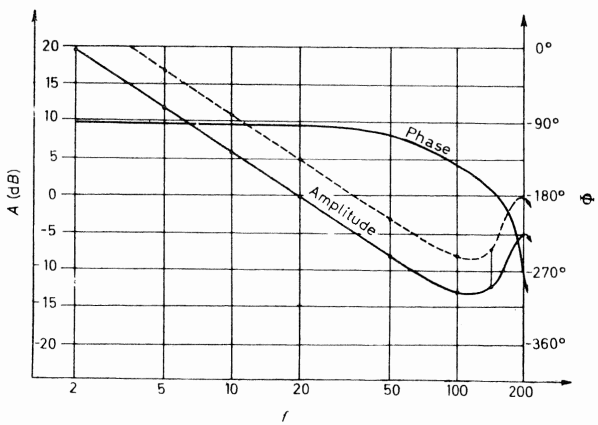


Figure 6.18. Frequency response curves

6.6. STABILITY

A system is unstable if an infinitely small variation of the input, or an infinitely small perturbation, causes variations of the output which are not infinitely small and which do not tend to disappear after the input variation or perturbation has been removed. (Mathematically, the oscillations are of infinite magnitude, but in practice they are limited only by saturations in the system such as zero absolute pressure or mechanical limits.)

An unstable system is generally unusable. Stability is therefore of major importance in any system.

6.6.1. LINEAR SYSTEMS. THE GENERAL CASE

A linear system is stable if the real part of each of the poles of its transfer function is negative.

A linear system has a transfer function in the form of a rational fraction which can be reduced to its components

$$\sum \frac{K_1}{p - \alpha_1} + \sum \frac{a\,p + b}{(p - \alpha_2)^2 + \beta^2}$$

giving the inverse transforms*

$$\sum K_1\, e^{\alpha_1 t} + \sum K_2\, e^{\alpha_2 t} \cos\,(\beta t + \Phi)$$

which tends to infinity only if α is positive.

6.6.2. LINEAR SYSTEMS WITH FEEDBACK

If a system consists of a succession of components in the form of an open chain, the analysis of its stability is straightforward.

The addition or modification of a component F_i does not affect the poles corresponding to the other components. The transfer function of the whole chain is the product of the transfer functions of its components.

This does not, however, apply to more complex systems and in particular to feedback systems. In this case, an action on a component F_i of the principal chain affects *all* the roots of the denominator of the *closed*-loop transfer function, since this has the form:

$$1 + F_1 F_2 F_3 \dots F_n$$

Despite the existence of methods for determining the sign of the roots of an algebraic equation (Routh's criterion), an analysis of the stability of the equation

$$1 + F_1 F_2 F_3 \dots F_n = 0$$

---

* K_2 and ϕ are constants given by

$$K_2 = \frac{1}{\beta} \sqrt{a^2\beta^2 + (b + \alpha_2 a)^2}$$

$$\varphi = -\tan^{-1} \frac{b + \alpha_2 a}{a\beta}$$

is complicated, and the effect of each component cannot be assessed clearly.

The condition for *stability of a feedback system* has been given by Nyquist in a most useful way, since it has the advantages of being easy to apply and of showing the effect of each component on the stability of the whole.

Consider a feedback signal of -1. Let $(TF)_0$ be the transfer function of the direct chain, known as the *open-loop transfer function*, and L_0 the corresponding transfer locus, known as the *open-loop transfer locus*. Nyquist's criterion states:

A closed system with a -1 feedback is stable if the number of times its open-loop transfer locus encircles the point -1 in an anticlockwise direction going from $\omega = -\infty$ to $\omega = +\infty$ is equal to the number of poles in the positive real half of the plane.

For a system with a stable open loop, as is generally the case in hydraulics, the criterion becomes:

If the open loop is stable (no pole in the real positive part), the closed system will be stable if the open loop transfer locus between $\omega = -\infty$ and $\omega = +\infty$ does not encircle the point -1.

Note that for the *Nichols plot*, which it is customary to use, in the general case corresponding to normal systems, the system is stable if the transfer locus intersects the phase axis on the right-hand side of the point 0 or if this point lies on the right-hand side of an observer moving along the locus in the direction of increasing ω (cf. *Figure 6.19*).

Stability margin—One would expect that the stability will become greater as the transfer locus moves away from the point 0 of the Nichols plot. We define

the *phase margin* $= 180° - |\phi_1|$ where ϕ_1 is the phase at unit gain, 0 dB

the *gain margin* $= -A_{180}$, where A_{180} is the amplitude in decibels at the phase angle $\phi = -180°$ (see *Figure 6.19*).

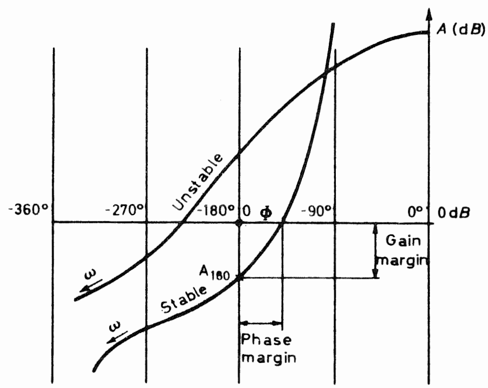


Figure 6.19

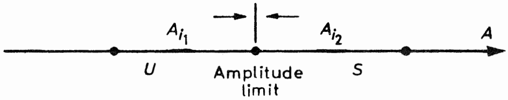
6.6.3. NON-LINEAR FEEDBACK SYSTEMS

The criteria given above can be applied to non-linear systems with linearizable equations. Systems might be imagined which remain stable for small perturbations but which diverge for perturbations so large that the linearized equations do not apply, but this rarely happens in practice. For non-linearizable equations analysed using the equivalent linear equations, it has been mentioned that the transfer function is valid only for inputs which are approximately sinusoidal and that the value of the transfer function depends on the input amplitude.

It is therefore necessary to consider an infinite number of transfer loci. If the condition for stability is satisfied by all these loci, the system will be stable.

If the condition for stability is satisfied by all loci corresponding to amplitudes greater than A_l, then we can predict the existence of an oscillation whose amplitude is approximately A_l.

Figure 6.20



If the condition for stability is satisfied by all loci corresponding to amplitudes less than A_d, we can predict the existence of a *dead zone of oscillation* approximately equal to A_d. (The system will only diverge if the initial impulse gives an amplitude greater than A_d).

If successive regions of stability and instability occur as the amplitude is increased, then a system which has an initial amplitude of oscillation A_i will amplify or damp these oscillations according to whether A_i is in the unstable or stable region (*Figure 6.20*).

This type of system will be investigated below (Section 6.10).

6.6.4. EXAMPLE: STABILITY OF A FUEL REGULATOR

We shall use the fuel regulator considered at the end of Section 6.4.4 as an example. Suppose that the loss of head through the valve is independent of the speed of rotation of the engine, as is the case in the majority of actual systems.

The block diagram (*Figure 6.10*) becomes that of *Figure 6.21*, since $\Gamma = 0$, or in canonical form, that of *Figure 6.22*.

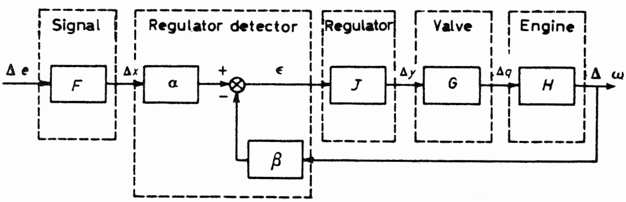


Figure 6.21

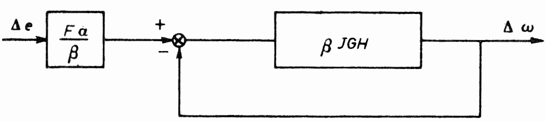


Figure 6.22

For reasons of accuracy which will be discussed later, the regulator very often has an integrating component with a transfer function K/p. To obtain this integration, all that is necessary is that when the detector is not in equilibrium, a hydraulic control valve is opened, supplying oil to the ram controlling the valve: the speed of opening of the valve dy/dt, corresponds to an error signal ϵ, so that $J = V/\epsilon = K/p$.

Since β, K and G are constants and $H = A/(1+Tp)$ (Section 6.4.4), the open-loop transfer function of the regulator is

$$H_0 = \frac{\beta K G A}{p(1 + Tp)}$$

The corresponding transfer locus is shown in *Figure 6.23.*

It can be seen that whatever the gain of the loop (detector gain β, regulator gain K or valve gain G), the transfer locus will not be on the left-hand side of point 0.

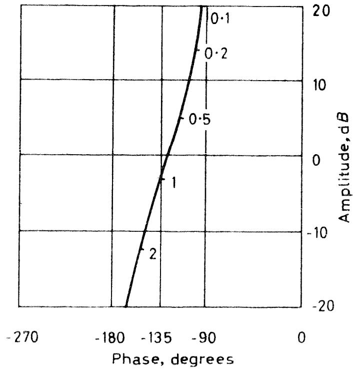


Figure 6.23. Curve of $F(j\omega) = 1/[Tj\omega(1=Tj\omega)]$, *graduated in* ωT

We might well conclude that such a regulator is always stable, although poorly damped if the gain is high. In fact, it can be established experimentally that it will then be *definitely unstable*—because the equation which was set up was only approximate. Clearances, friction forces and inertia effects (both of the fuel and the mechanical parts) cause the regulator to respond with an additional lag that has not been taken into account.

To be more accurate, we should have put

$$H_0 = \frac{\beta K G A}{p(1 + Tp)(1 + \tau p)}$$

Rather than going to some lengths to determine this time lag, it is preferable to use a *phase margin* (for example: $\phi_M = 45°$). The maximum allowable total gain can immediately be determined: all that is necessary is to displace the Nichols plot *vertically* until it cuts the phase axis at a point $180° - \phi_M$ ($135°$ for $\phi_M = 45°$).

In the very simple case given, the determination of this gain can be made just as rapidly by calculation. We can write H_0 in the form

$$H_0 = \beta K G A T \frac{1}{Tp(1 + Tp)}$$

For $\omega = 1/T$:

$$\frac{1}{1 + Tp} = \frac{1}{1 + i} = \frac{1}{\sqrt{2}}(\cos 45° - i \sin 45°)$$

The total phase displacement is 135°, and the amplitude, $1/\sqrt{2}\beta KGAT$, should be equal to, or less than, 1. By putting the terms which can be altered by the manufacturer of the regulator on the left-hand side, we have

$$\beta KG \leqslant \frac{\sqrt{2}}{AT}$$

In practice, this condition is incompatible with the response speed required, so that it is necessary to introduce a compensating system in order to increase the gain without the danger of instability.

Without considering the details, we shall accept that a convenient compensating system (see Section 10.8) has a transfer function of

$$C = \frac{1}{20}\,\frac{1 + Tp}{1 + Tp/20}$$

so that

$$H_0' = H_0 C = \frac{\beta KGA}{p(1 + Tp)}\,\frac{1}{20}\,\frac{1 + Tp}{1 + Tp/20} = \frac{\beta KGAT}{400 \cdot Tp/20\,(1 + Tp/20)}$$

The condition for stability is therefore

$$\beta KG \leqslant \frac{400\sqrt{2}}{AT}$$

Thus the gain of the open loop can be increased by a factor of 400 for the same stability margin and at the same time, due to the attenuation of 20 introduced by the compensating system, the response speed of the regulator is increased by a factor of 20.

6.7. RESPONSE SPEED

6.7.1. WHAT IS THE RESPONSE SPEED OF A SYSTEM?

In a system whose function is to convert a given input to an output, it would be ideal to have an infinite response speed, i.e. for the output to follow the input without any lag. This performance cannot be achieved in practice since, for example, it would require an infinite force acting on an element of given mass.

In general, the response to an input of given amplitude and duration reaches its ultimate value asymptotically.

To assess the response speed, we define a *response time*, i.e. *the time which elapses between the commencement of a certain input signal and the moment when the output reaches n per cent of its final value.*

To be more precise, we define a *time constant* (characteristic time) as the time taken for the output to reach the fraction $1-(1/e)$ (about $\frac{2}{3}$) of its final value, for a step input (see Section 6.2.4.). This definition has been adopted because of its usefulness for first-order systems with which, in a first approximation, we can often compare more complex systems.

The transfer function of a first-order system

$$\frac{S}{E} = \frac{K}{1 + tp}$$

has a time constant of t exactly.

If we are particularly interested in the transmission of sinusoidal signals, the consideration of the time constant is replaced by that of the *cut-off frequency*.

In general, as the frequency of a sinusoidal input is increased, the phase of the output is displaced and its amplitude reduced. The cut-off frequency is that at which either the phase displacement reaches ϕ_0 degrees or the amplitude is reduced to A_0 decibels.

When discussing the cut-off frequency, it is essential to state whether the primary interest is in attenuation or phase displacement and to give the respective reference value.

Electronic engineers often use a cut-off frequency corresponding to 3 dB; this definition is also useful for first-order systems whose cut-off frequency at 3 dB is

$$f_c = \frac{1}{2\pi t}$$

since then

$$\frac{S}{E} = \frac{K}{1+j} \quad \text{and} \quad \left|\frac{S}{E}\right| = \frac{K}{\sqrt{2}}$$

The cut-off angular frequency is

$$\omega_c = 1/t = 2\pi f_c$$

For servo controls, the phase displacement is often more critical than the damping and it is usual to refer to a cut-off frequency at 45°. (Note that this frequency is equal to $1/2\pi t$ for a first-order system, but there is no reason for it to equal the cut-off frequency at 3 dB for a system which is not first order). When the transfer function of a hydraulic system is of the second order, it is preferable to use a cut-off frequency corresponding to a phase angle of 90°.

For a system having a second-order transfer function

$$\frac{S}{E} = \frac{K}{1 + 2\zeta \dfrac{p}{\omega_0} + \dfrac{p^2}{\omega_0^2}}$$

this cut-off frequency is equal to $\omega_0/2\pi$.

So far, we have defined response time and cut-off frequency without giving the input amplitude; this is justifiable for the linear case. In the course of normal operation, however, amplitudes are frequently encountered which are sufficiently high for one of the components of the system to become saturated. In this case, the response time and cut-off frequency are changed considerably.

For a real system, therefore, and in particular for a hydraulic system, the statement of the response time and cut-off frequency should be accompanied by giving the corresponding maximum amplitude. In practice it is usual to specify two response times or two cut-off frequencies for two different amplitudes. For example, an aircraft manufacturer may require a servo control with

$$\text{cut-off frequency at } 90^\circ \begin{cases} \geqslant 10 \text{ c/s} & \text{for} \quad e \leqslant e_{\max}/10 \\ \geqslant 2 \text{ c/s} & \text{for} \quad e = e_{\max} \end{cases}$$

6.7.2. DETERMINATION OF THE RESPONSE SPEED NECESSARY IN A SYSTEM

The selection of the response speed of a system is a complicated problem. First of all, there is no point in increasing the response speed indefinitely because

(1) this requires more energy and therefore increases volume, weight and price;
(2) too high a response speed introduces the risk of the transmission of high-frequency perturbations which are associated with all physical quantities ('noise'); in certain systems having a high degree of amplification this noise may even completely saturate the amplification stages and thus stifle all useful signals;
(3) too high a response speed is often dangerous for the control components.

We should therefore not exceed the response speed *necessary* for satisfactory operation. This necessary speed is basically determined by two requirements:
(1) for the correct functioning of the system itself, certain signals have to be transmitted in a certain time.
(2) a system which is a component of a more complex system must respond with a speed sufficient to preserve the stability of the whole.

The first condition means the requirement of a certain response time to a high-amplitude input, the second that of a certain cut-off frequency for low-amplitude inputs.

Example 1—The specification for the servo control mentioned above included the requirement that the aircraft in which it was fitted should be able to be put into a tight turn within a given time. This led to the requirement of a cut-off frequency of 2 c/s at 90° for $e = e_{\max}$. On the other hand, to ensure stability of the flight path of an aircraft equipped with an automatic pilot, it is necessary that inputs of small amplitude ($e \leqslant e_{\max}/10$) be transmitted with a response speed corresponding to a cut-off frequency of 10 c/s.

Note that a servo control with a cut-off frequency of 10 c/s for $e = e_{\max}$ would be much larger and heavier, would use much more energy and could cause physical failure of the aircraft by putting it into flight paths incompatible with its structural strength.

Example 2—The fuel regulator used as an example in previous Sections should have a response time of several hundredths of a second for low-amplitude signals. On the other hand, it would be dangerous for the operation of the engine if it responded to an abrupt movement of the manual control, initially in the idling position, in less than 5 or 6 sec.

In practice, when regulation is achieved by varying the displacement volume of a variable flow pump, the response time to low-amplitude signals is of the order of $\frac{1}{100}$ sec (this is the response time of the hydraulic potentiometer and ram assembly which operates the inclined cam controlling the displacement volume). But the time taken to increase the flow from zero to maximum is 1–2 sec, since the hydraulic potentiometer is operating at its mechanical limit (saturation).

In the complete installation, the increase of fuel flow is slowed down even further by a special component, known as an acceleration controller, so that the build-up time is increased to the 5 or 6 sec necessary to avoid overheating.

6.7.3. DETERMINATION OF RESPONSE SPEED

The methods used for calculating the response speed of a system depend on whether:

the input amplitude is large or small;

the input is sinusoidal or non-sinusoidal.

6.7.3.1. *Low-amplitude sinusoidal inputs*

The phase displacement and attenuation of the output can be found immediately from the transfer function, by reading them directly off the frequency response curves. This simplicity of determining cut-off frequencies from the transfer function causes their frequent use as performance criteria, despite the artificial nature and restrictions caused by having to refer to sinusoidal inputs.

6.7.3.2. *Low-amplitude non-sinusoidal inputs*

The calculation is theoretically simple. Since the Laplace transform of the output is equal to the product of the Laplace transform of the input and the transfer function, there is no theoretical difficulty in finding the output (inverse transformation).

In practice, these calculations are often tedious and they are seldom tackled these days without the use of an analogue computer. Alternatively, it is possible to obtain an approximation by replacing the transfer function by a simpler function which approximates to it at low frequencies.

6.7.3.3. *High-amplitude inputs*

'High-amplitude input' is taken to mean inputs which cause at least one component of a system to *saturate for an appreciable time.*

Accurate estimation of the output is fairly difficult without the use of a computer (and even then only the more expensive machines can be adapted to take saturations into account). Here again, however, an approximate estimation is fairly simple. This will be illustrated briefly by an example.

Example—Estimation of the flow from a variable-flow pump as a function of the signal to the potentiometer controlling the ram of the inclined cam—It has been mentioned above that the potentiometer (cf. example 2, Section 6.7.2) is very easily saturated. This potentiometer controls the servo flow to the ram, so that its saturation corresponds to saturation of the displacement velocity of the cam and therefore of the second derivative of the flow of fuel supplied by the pump.

If we consider a step input considerably larger than that at which saturation takes place, we may assume that the control ram is displaced at constant velocity and that the second derivative of the fuel flow is therefore constant.

Figure 6.24 shows the development of the flow with time for step inputs of different magnitudes. It illustrates, in particular, that the approximation of constant velocity for large-input amplitudes is easy to use and is preferable to that neglecting saturation. This sort of approximation is useful in many cases, and its use may well prevent serious mistakes.

When dealing with a sinusoidal input, we must first check that the linear estimation leads to a maximum flow variation $(\mathrm{d}Q/\mathrm{d}t)_M$ greater than the flow variation corresponding to saturation of the potentiometer $(\mathrm{d}Q/\mathrm{d}t)_s$. If $(\mathrm{d}Q/\mathrm{d}t)_M$ is definitely greater than $(\mathrm{d}Q/\mathrm{d}t)_s$, the following approach may be used (*Figure 6.25*). Let C_1 be the curve representing the linear solution for the flow. The true curve C_2 consists of a number of straight lines of slope $\pm\,(\mathrm{d}Q/\mathrm{d}t)_s$ in the form of a saw-tooth.

If the input frequency is f (period, $T = 1/f$), the upper limit of the output is Q_s, where

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_s = 4\,f\,Q_s\,,$$

so that

$$Q_s = \frac{(\mathrm{d}Q/\mathrm{d}t)_s}{4\,f}$$

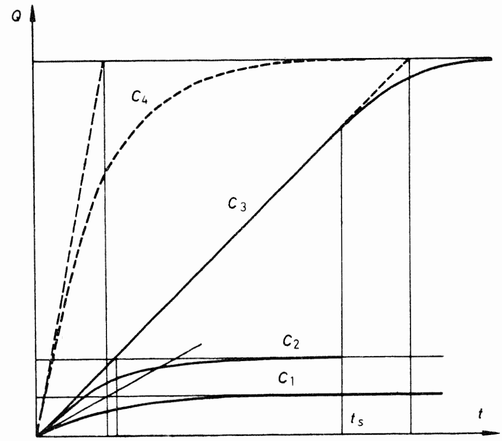


Figure 6.24. Curve C_1: very-low-amplitude input (linear operation)
Curve C_2: higher-amplitude input (saturation reached)
Curve C_3: high-amplitude input (system in saturation up to time t_s)
Curve C_4: virtual response to same input in absence of any saturation

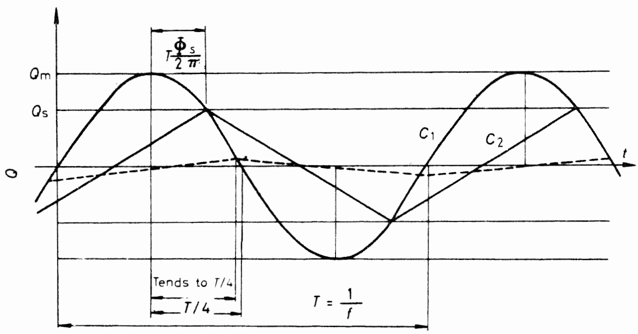


Figure 6.25

An approximate value (an underestimation) of the supplementary phase lag introduced by saturation is given (see *Figure 6.25*) by

$$\phi_s = \cos^{-1}\frac{Q_s}{Q_M}$$

(ϕ_s tends to $\pi/2$ as Q_s/Q_M tends to zero).

6.7.4. INCREASING THE RESPONSE SPEED

For high-amplitude inputs which cause a component to operate under saturation conditions, the response speed can obviously be improved by increasing the saturation level, for example by enlarging the orifices of the hydraulic potentiometer in the example given above.

For low-amplitude inputs, one might improve the response speed of individual components, at least of the slower ones; in a servo system, increase the gain of the open chain: the benefit from this operation is obvious from the Nichols chart, but so is the corresponding deterioration in stability (cf. *Figure 6.23*). In Chapter 10 we shall consider several processes used in hydraulics to give better solutions to this problem.

6.8. ACCURACY

Errors in the output can be caused either by imperfections of various components of the system or by the introduction of parasitic inputs known as perturbations.

The error is clearly a function of time. It is usual, however, to consider the error in the final condition, i.e. the error which persists a sufficiently long (theoretically, an infinite) time after the introduction of a step input.

This determination can therefore be classified under the heading of 'static analysis'. It can be made graphically on the characteristic curves given in Section 3.7.

However, certain results of dynamic analysis produce useful generalizations and simplifications, as we shall see below.

6.8.1. IMPROVING THE ACCURACY BY SERVO CONTROL

Consider an open-loop system, i.e. a system consisting of components connected in series, each affecting the following one.

If the gain, F_0, of a component changes by n per cent, a relative error of n per cent is obviously introduced to the output signal. Now it is difficult to make power amplifiers having a constant gain. Open-loop power-amplifying systems cannot therefore be very accurate.

To improve the accuracy, we must therefore separate the power-amplifying components from those giving accuracy.

This is achieved in feedback or servo systems in which the direct chain provides the power and the feedback (or detection) chain ensures accuracy. To determine the extent to which this separation of functions is achieved, consider a linear servo system in which the transfer functions of the direct chain and the feedback chain are, respectively (*Figure 6.26*)

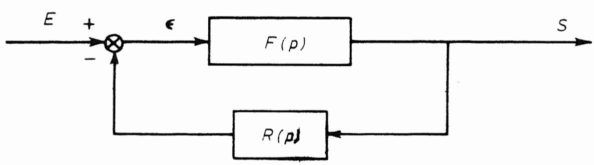


Figure 6.26

$$F(p) = F_0 \frac{1 + ap + \cdots}{1 + bp + \cdots} \qquad R(p) = R_0 \frac{1 + cp + \cdots}{1 + dp + \cdots}$$

The closed-loop transfer function is

$$\frac{S}{E} = \frac{F(p)}{1 + R(p)F(p)}$$

The response to a step input e_0* is

$$S = \frac{e_0}{p} \frac{F(p)}{1 + R(p)\,F(p)}$$

and the final value of S (cf. Section 6.2.3)

$$s_\infty = \lim_{p \to 0} \frac{e_0}{p} \frac{p\,F(p)}{1 + R(p)\,F(p)} = e_0 \frac{F_0}{1 + R_0 F_0}$$

i.e.

$$s_\infty = \frac{e_0}{R_0} \frac{1}{1 + 1/R_0 F_0} \tag{12}$$

A relative error of $\Delta F_0/F_0$ in the gain of the power chain corresponds to an output error of

$$\frac{\Delta s}{s_\infty} = \frac{1}{1 + R_0 F_0} \frac{\Delta F_0}{F_0} \dagger$$

If the open-loop gain, R_0F_0, is much greater than 1, we can write

$$\boxed{\frac{\Delta s}{s_\infty} = \frac{1}{R_0 F_0} \frac{\Delta F_0}{F_0}} \tag{13}$$

or

$$\boxed{\Delta s = \frac{e_0}{R_0} \frac{1}{R_0 F_0} \frac{\Delta F_0}{F_0}} \tag{14}$$

---

* In fact, $\mathscr{L} e_0 = e_0/p$; cf. Section 6.2.4.

† Since

$$\frac{\Delta s}{s_\infty} = -\frac{\Delta(1 + 1/R_0F_0)}{1 + 1/R_0F_0}$$

and

$$\Delta\left(1 + \frac{1}{R_0F_0}\right) = \Delta\left(\frac{1}{R_0F_0}\right) = -\frac{1}{R_0F_0} \frac{\Delta F_0}{F_0}$$

we have

$$\frac{\Delta s}{s_\infty} = +\frac{\Delta F_0}{F_0} \frac{1/R_0F_0}{1 + 1/R_0F_0} = \frac{\Delta F_0}{F_0} \frac{1}{1 + R_0F_0}$$

In a linear system, the output error due to inaccuracy of the gain of the power chain can theoretically be decreased by increasing the open-loop gain. (In practice, as we saw, the increase in gain is limited by the risk of instability.)

Important notes

(1) Referring to eqn. (12), it can be seen that if R_0F_0 is much greater than unity, a *relative error* of n per cent in R_0 introduces an equal output error.

In a servo system with a large open-loop gain, all errors in the feedback chain introduce an equal relative error to the output.

In practice, therefore, it is very often the accuracy of the feedback chain which determines the overall accuracy of a servo system. This fact is often forgotten and much time and effort spent in decreasing errors due to variations of the gain of the direct chain or to perturbations, even though they are already very much less than those introduced by the detectors.

In certain cases, however, a variation of the gain of the feedback chain can introduce a very much smaller, or even zero, output error. For example, a position servo system in which the error is the potential difference between the sliders of a control potentiometer (input) and a servo-potentiometer (feedback), is practically insensitive (from the point of view of accuracy) to variations of the supply voltage, provided that the potentiometers have a common supply.

(2) Consider a servo system with *an integration in its direct chain*

$$H(p) = \frac{H_0}{p}\,\frac{1 + ap + \cdots}{1 + bp + \cdots}$$

being the open-loop transfer function. This time the final value of the output is

$$s_\infty = \lim_{p \to 0} \frac{e_0}{p}\, p \,\frac{\dfrac{H_0}{p}\,\dfrac{1 + ap + \cdots}{1 + bp + \cdots}}{1 + R_0\,\dfrac{1 + cp + \cdots}{1 + dp + \cdots}\,\dfrac{H_0}{p}\,\dfrac{1 + ap + \cdots}{1 + bp + \cdots}} = \frac{e_0}{R_0}$$

The output error due to a variation in the open-chain gain is zero for a step input. This result is obvious if we note that integration signifies infinite gain in position F_0.

6.8.2. OUTPUT ERROR OF A SERVO MECHANISM

For servomechanism engineers, a servo system is a system whose function is to nullify the quantity $\epsilon = E - RS$, which is known as the *input error* or usually just the *error*.

When ϵ is zero,

$$S = \frac{E}{R}$$

and in the final condition

$$s_\infty = \frac{e_0}{R_0}$$

For this reason, servomechanism engineers define the output error* by the equation

$$\boxed{\Delta S = \frac{E}{R} - S}$$

Thus, in the final condition

$$\Delta s = \frac{e_0}{R_0} - s_\infty = \frac{e_0}{R_0}\left[1 - \frac{1}{1 + \dfrac{1}{R_0 F_0}}\right] = \frac{e_0}{R_0}\,\frac{1}{1 + R_0 F_0}$$

and if $R_0 F_0 \gg 1$:

$$\boxed{\Delta s = \frac{e_0}{R_0}\,\frac{1}{R_0 F_0}} \qquad (15)$$

or

$$\boxed{\frac{\Delta s}{s_\infty} = \frac{1}{R_0 F_0}} \qquad (18)$$

i.e. eqn. (13) and (14) without the $\Delta F_0/F_0$ term.

Thus the *output error*, in the servomechanism sense, is equal to the real error introduced by a *unit* relative variation of the gain of the direct chain. Its use in the estimation of the accuracy of a system generally gives a pessimistic result since $\Delta F_0/F_0$ rarely† reaches 1.

6.8.3. ERROR DUE TO PERTURBATIONS

6.8.3.1. *Perturbations introduced into the direct chain*

Suppose a perturbation, D, is introduced between components F and G of the direct chain (*Figure 6.27*). In Section 6.4.2 it was shown that the output is

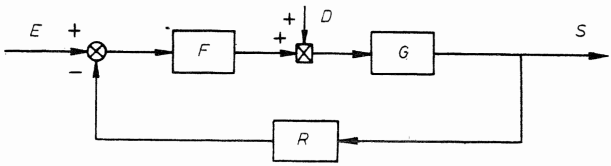


Figure 6.27

* For systems with unit feedback, to which real systems are very often reduced, *the input and output errors are equal* because $R = R_0 = 1$.

† There are cases where these two errors are the same. We will find examples of this in Chapter 7 when considering stabilizing devices for servo controls. These have transfer functions in which the gain of the direct chain is inversely proportional to the external stiffness opposing the ram (and therefore infinite when this stiffness is zero). In these conditions, between zero and finite stiffness, $\Delta F_0/F_0 = 1$.

$$S = S_E + \Delta S_D = E\frac{FG}{1 + RFG} + D\frac{G}{1 + RFG}$$

The second term, ΔS_D, is the error due to the perturbation. The final value for a step perturbation, d_0, is

$$\Delta s_D = d_0\frac{G_0}{1 + R_0 F_0 G_0} = d_0\frac{1}{R_0 F_0 + 1/G_0}$$

If $R_0 F_0 \gg 1/G_0$ (this is generally true), this becomes

$$\Delta s_D \simeq d_0\frac{1}{R_0 F_0}.$$

If the component F is an *integrator* ($F_0 = \infty$), $\Delta s_D = 0$. If the component G is an *integrator* and F is not, $\Delta s_D \neq 0$.

This result is the basis of the following important theorem:

In order that a perturbation in the direct chain shall not result in an output error, it is necessary and sufficient that it be introduced downstream of an integration.

Note 1—There are two other cases where Δs_D is theoretically zero: (*a*) where R is an integrator (R_0 infinite); (*b*) where G is a differentiator ($G_0 = 0, 1/G_0$ infinite). These cases are of no real interest, since they correspond to a system giving zero output to a step input, as we can see immediately by writing the first term of S in the form

$$S_E = E\frac{1}{R + 1/FG}$$

Note 2—To remember the position of the integrator relative to the perturbation for zero output error, the following mnemonic aids may be used. A variation of the input obviously results in a variation of the output. A perturbation which is similar to an input, i.e. which is *not separated from the input by an integration*, results in a finite error.

6.8.3.2. *Perturbations introduced into the feedback chain*

Consider the system (*Figure 6.28*)

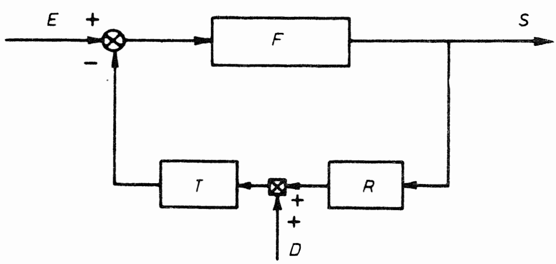


Figure 6.28

$$S = S_E + \Delta S_D = E\,\frac{F}{1 + RTF} + D\,\frac{TF}{1 + RTF}$$

so that

$$\Delta S_D = D\,\frac{1}{R + 1/TF}$$

and the final value is

$$\boxed{\Delta s_D = d_0\,\frac{1}{R_0 + 1/T_0 F_0}}$$

If $R_0 \gg 1/T_0 F_0$ (which is generally the case)

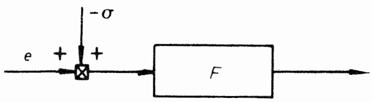
$$\boxed{\Delta s_D \simeq d_0\,\frac{1}{R_0}}$$

In practice this error is never zero, since if R_0 were infinite or F_0 zero, the servo system would have zero output.

6.8.4. ERRORS DUE TO A DEAD ZONE IN DIFFERENT COMPONENTS

Let σ be the limit of a dead zone of a component. For an input equal to σ, the output is zero. For inputs greater than σ, we can replace the real component by a perfect component with an input e and a perturbation $-\sigma$ (*Figure 6.29*).

Figure 6.29



The output error due to the dead zone is the variation Δs of the output which, all other things being equal, is produced by a variation σ of the input to the component considered. Note that *an integration will not nullify errors due to dead zones in components upstream* (cf. Chapter 10, Example 7, Section 6).

6.9. LACK OF NATURAL DAMPING IN HYDRAULIC SYSTEMS

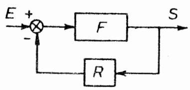
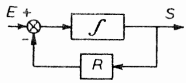
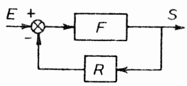
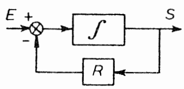
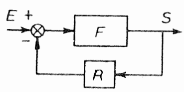
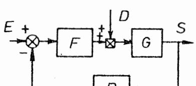
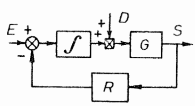
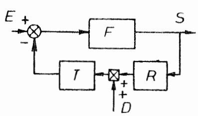
In the analysis of hydraulic systems, second-order differential equations of the following form are often encountered

$$y = rx + m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

i.e. without a damping term in $\mathrm{d}x/\mathrm{d}t$. If the component under consideration is a non-active one, then the equation may be erroneous and the analysis should be checked. This may still produce no $\mathrm{d}x/\mathrm{d}t$ term, so that we are left with an equation of the form given above. In fact,

(1) for systems which are not hydraulic, the equation usually includes a damping term in $\mathrm{d}x/\mathrm{d}t$ related to known quantities which are fixed and measurable, such as the resistance of an electrical circuit. In hydraulic systems,

Table 6.5. Summary

Error	*Block diagram*	*Exact value*	*Approximate value*
Output error (as defined by servo mechanism engineers)		$\Delta s = \dfrac{e_0}{R_0}\,\dfrac{1}{1 + R_0 F_0}$	$\Delta s = \dfrac{e_0}{R_0}\,\dfrac{1}{R_0 F_0}$ $(R_0 F_0 \gg 1)$
		$\Delta s = 0$	
Error due to *variation of gain* of an element of the *direct chain*		$\Delta s_F = \dfrac{e_0}{R_0}\,\dfrac{1}{1 + R_0 F_0}\,\dfrac{\Delta F_0}{F_0}$	$\Delta s_F = \dfrac{e_0}{R_0}\,\dfrac{1}{R_0 F_0}\,\dfrac{\Delta F_0}{F_0}$ $(R_0 F_0 \gg 1)$
		$\Delta s_F = 0$	
Error due to *variation of gain* of an element of the *feedback chain*		$\Delta s_R = -\dfrac{e_0}{R_0}\,\dfrac{R_0 F_0}{1 + R_0 F_0}\,\dfrac{\Delta R_0}{R_0}$	$\Delta s_R = -\dfrac{e_0}{R_0}\,\dfrac{\Delta R_0}{R_0}$ $(R_0 F_0 \gg 1)$
Error due to a *perturbation* in the *direct chain* (or to a dead zone in an element of this chain)		$\Delta s_D = d_0\,\dfrac{1}{1/G_0 + R_0 F_0}$	$\Delta s_D = d_0\,\dfrac{1}{R_0 F_0}$ $(R_0 F_0 \gg 1/G_0)$
		$\Delta s_D = 0$	
Error due to a *perturbation* in the *feedback chain* (or to a dead zone in an element of this chain)		$\Delta s_D = d_0\,\dfrac{1}{1/T_0 F_0 + R_0}$	$\Delta s_D = d_0\,\dfrac{1}{R_0}$ $(R_0 \gg 1/T_0 F_0)$

Note: Remember that it is not possible to alter R_0, which is determined by the ratio of output to input required.

however, the damping terms are usually functions of quantities which are difficult to estimate, such as internal leakage flows. The value of these terms is often small, so as not to affect the performance (e.g. flow dissipated through leakage); moreover, *it varies*: *for different components*, since it is difficult to fix the leakage flow with any accuracy, especially if it is mainly due to imperfections and tolerances of manufacture; *with time* in any component as it becomes worn; *with conditions*, i.e. essentially with the type of liquid used and its temperature; *with amplitude*, since the leakage coefficient, expressed as flow per unit pressure difference, does not remain constant when the flow becomes turbulent. The equation is therefore no longer linear, and in practice we would have to consider a number of linear equations approximating to the true equation for different values of the amplitude.

This is the reason for the bad reputation of hydraulic systems, which are accused of 'not agreeing with theoretical estimates' or of 'always having surprises in store'.

(2) *Coulomb friction* (this expression is used in preference to the misleading term 'dry friction') is difficult to introduce into the equations and in any case has a coefficient of damping which *is small* and *varies: for different components*, since it is a function of clearance and alignment; *with time*, owing to the change of surface finish and hardness of fittings; *with conditions of use*, i.e. not only on the type of liquid but also on the previous operations or stoppages (wetness, stiction and stiffness at the commencement of operation); *with amplitude*, as we shall see later.

Insufficient damping is a characteristic fault of hydraulic systems. The introduction of artificial damping is generally expensive, owing either to its technological complexity or to the quantity of energy consumed, e.g. as in leakage flow. It should therefore be used only as a last resort and then strictly limited to the minimum required.

This does not mean that hydraulic systems should be condemned. We must accept the fact that they have low damping coefficients and proceed accordingly, paying particular attention to the following points, which are not necessarily listed in the chronological order in which they should be considered:

(1) *increasing the resonant frequency*, so that it is as far above the maximum operating (cut-off) frequency as possible;
(2) *estimating the natural damping*, and in particular the lower limit of damping, which can be expected;
(3) *devising a compensator system* which can be thought of either as artificial damping, as a low-pass filter adjusted between the cut-off frequency and the resonant frequency, or as a phase-modifying system which moves the resonance point away from the 180° phase.

It does not seem possible to give a general method to follow in each case, so the above considerations are not pursued in detail but merely given as examples. An exception is the following Section which deals with Coulomb friction. Its introduction into an analysis is generally difficult, but it becomes relatively simple for low damped systems near the resonant frequency, providing certain approximations, which are physically justified, are made.

6.10. INTRODUCTION OF COULOMB FRICTION INTO THE ANALYSIS OF POORLY DAMPED SYSTEMS

6.10.1. SECOND-ORDER SYSTEM WITH NO EXTERNAL FORCES

Consider a second-order system of displacement z, mass m and stiffness r. Let the Coulomb friction be f.

If the symbol $[\sigma]$ has the values

$$\begin{cases} +1 & \ldots \text{ for } \dfrac{dz}{dt} > 0 \\ -1 & \ldots \text{ for } \dfrac{dz}{dt} < 0 \\ -1 < [\sigma] < +1 \text{ (actual value determined by the equilibrium equation)} & \ldots \text{ for } \dfrac{dz}{dt} = 0 \end{cases}$$

then the equation of motion (second-order differential equation with no second term) is

$$rz + f \cdot [\sigma] + m \frac{d^2z}{dt^2} = 0$$

or

$$rz + \frac{f \cdot [\sigma]}{dz/dt} \cdot \frac{dz}{dt} + m \frac{d^2z}{dt^2} = 0 \tag{1}$$

If the maximum value $\dfrac{f \cdot [\sigma]}{(dz/dt)_{max}}$

of the positive quantity $\varphi(t) = \dfrac{f \cdot [\sigma]}{(dz/dt)}$

is small compared with $\sqrt{mr}$, it can be shown (see Appendix 6.1) that the motion defined by eqn. (1) is very similar to that given by the *linear* equation

$$rz + \varphi_0 \frac{dz}{dt} + m \frac{d^2z}{dt^2} = 0 \tag{2}$$

Putting $\omega_0 = \sqrt{r/m}$ and $\zeta_0 = \psi_0\omega_0/2r = \psi_0/2\sqrt{mr}$ this becomes

$$z + \frac{2\zeta_0}{\omega_0} \frac{dz}{dt} + \frac{1}{\omega_0^2} \frac{d^2z}{dt^2} = 0 \tag{2'}$$

where ψ_0 represents the viscous friction absorbing the same energy per cycle* as the Coulomb friction, f, in the actual motion.

* In each case the interval between, for example, two successive maxima is called a cycle.

Eqn. (2) is the equivalent linear equation corresponding to the approximation of equivalent energy given in Section 5.6.

Estimation of ψ_0 *and* ζ_0—To evaluate ψ_0 and ζ_0, it is simply necessary to equate the expressions for the energy absorbed per cycle in the two motions. It is assumed that the motion is lightly damped and that the actual motion $z = f(t)$ is similar to $z = z_0 \sin \omega_0 t$.

(*a*) *Energy absorbed in the actual motion,* W_f:

$$W_f = 4\int_0^{z_0} f \mathrm{d}z = 4fz_0$$

(*b*) *Energy absorbed in the approximate motion,* W_a

$$W_a = 4\int_0^{z_0} \psi_0 \frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}z$$

and since $\mathrm{d}z/\mathrm{d}t = z_0\omega_0 \cos \omega_0 t$,

$$W_a = 4\int_0^{\pi/2} \psi_0 z_0^2 \omega_0 \cos^2 \omega_0 t \,.\, \mathrm{d}(\omega_0 t) = \pi\psi_0 z_0^2 \omega_0$$

Equating W_f and W_a

$$\psi_0 = \frac{4}{\pi}\frac{f}{z_0\omega_0} \tag{3}$$

and since $\zeta_0 = \psi_0\omega_0/2r$

$$\boxed{\zeta_0 = \frac{2}{\pi}\cdot\frac{f}{rz_0}} \tag{4}$$

Summing up, the true motion determined by the equilibrium between a restoring force, an inertia force and a Coulomb friction force can be made to correspond to a motion defined by a linear second-order equation in which *the damping term decreases when the amplitude is increased.*

Note—The above analysis, while not mathematically rigorous, is very useful in practice. The good agreement between the real and approximate motions for small values of ζ_0, i.e. for small values of f/rz_0, together with the deterioration of this agreement as ζ_0 increases, is illustrated in Appendix 6.1 which gives the phase trajectories of the two motions.

6.10.2. SECOND-ORDER SYSTEM WITH SINUSOIDAL EXTERNAL FORCE

Suppose that an external sinusoidal force $F = F_0 \sin \omega t$ is applied to the preceding system, resulting in a *forced vibration of periodic time* $2\pi/\omega$ when the motion is established.

The equation of motion is

$$F_0 \sin \omega t = rz + \frac{f \cdot [\sigma]}{\mathrm{d}z/\mathrm{d}t} \frac{\mathrm{d}z}{\mathrm{d}t} + m \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} \tag{5}$$

Noting that ω is, in general, different from $\omega_0 = \sqrt{r/m}$, we proceed as in Section 6.10.1, i.e. by replacing

$$\psi(t) = \frac{f \cdot [\sigma]}{\mathrm{d}z/\mathrm{d}t}$$

by the constant ψ_0, so that the energy absorbed per cycle is the same.

The evaluation of ψ_0 is the same as before, except that ω replaces ω_0 in the expression for z. Thus we have

$$\psi_0 = \frac{4}{\pi} \frac{f}{z_0 \omega} \tag{6}$$

and since $\zeta_0 = \psi_0 \omega_0 / 2r$,

$$\boxed{\zeta_0 = \frac{2}{\pi} \frac{f}{r z_0} \frac{\omega_0}{\omega}} \tag{7}$$

The *damping coefficient* of the approximate motion is *inversely proportional* not only to the *amplitude* but also to the *frequency*.

Notes—(1) We can derive a simple physical interpretation from this fact. The viscous friction proportional to velocity gives a constant damping coefficient while the Coulomb friction, being constant, will give a damping coefficient inversely proportional to the mean value of the velocity, i.e. proportional to $1/z_0\omega$.

(2) The describing function approach gives exactly the same linear equation, the first harmonic of the square function $f.[\sigma]$,

$$\frac{4}{\pi} \cdot f \cdot \cos(\omega t + \varphi)$$

being equal to the $\mathrm{d}z/\mathrm{d}t$ term of the equivalent linear equation:

$$r \frac{2\zeta_0}{\omega_0} \frac{\mathrm{d}z}{\mathrm{d}t} = r \frac{2\zeta_0}{\omega_0} z_0 \omega \cos(\omega t + \varphi)]$$

(3) The fact that ζ_0 depends on z_0 and on ω detracts from the value of the concept of the damping coefficient. It is better to approach the problem using the transfer function

$$\frac{z}{F} = \frac{1}{r\left(1 + \frac{4}{\pi} \frac{f}{r z_0} \frac{p}{\omega} + \frac{p^2}{\omega_0^2}\right)} \tag{8}$$

and the corresponding transfer locus, whose polar coordinates are therefore

$$\left\{\begin{aligned} A &= \frac{1}{r}\,\frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\left(\frac{4}{\pi}\,\frac{f}{r z_0}\right)^2}} \\ \varphi &= \tan^{-1}\left(\frac{\frac{-4}{\pi}\,\frac{f}{r z_0}}{1-\frac{\omega^2}{\omega_0^2}}\right) \end{aligned}\right. \tag{9}$$

For $\omega \gg \omega_0$, these equations are of no interest; all transmission is suppressed, whether the damping is viscous, Coulomb or zero and $A \to 0$ as $\phi \to -\pi$ (inertia effect).

For ω in the neighbourhood of ω_0, the results can be established using the concept of approximate motion: ϕ moves rapidly from $-\epsilon$ to $-\pi+\epsilon$ while A reaches a maximum absolute value of $\frac{1}{4}\pi\, z_0/f$. For $\omega \ll \omega_0$, it can be shown that, as $\omega/\omega_0 \to 0$,

$$\phi \to \tan^{-1}\left(-\frac{4}{\pi}\frac{f}{r z_0}\right)$$

and not to zero, as is the case for viscous friction. Likewise

$$A \to \frac{1}{r}\,\frac{1}{\sqrt{1+\left(\frac{4}{\pi}\,\frac{f}{r z_0}\right)^2}}$$

and not to $1/r$.

This result is very important: *The phase displacement of a second-order system with Coulomb friction does not tend to zero with the frequency, but to a limit which is inversely proportional to the amplitude.*

This also has a simple physical interpretation. Let $z^\star = F/r$ be the motion at low frequency in the absence of Coulomb friction (effect of mass negligible). If there is Coulomb friction, $F = rz+f[\sigma]$,

$$z = \frac{F}{r} - \frac{f}{r}[\sigma] = z^\star - \frac{f}{r}[\sigma]$$

It is easy to plot z starting from $z^\star$ (*Figure 6.30*).

As $z^\star$ increases, z follows at a distance f/r behind (curve $A^\star B^\star$ of $z^\star$ and AB of z). When $z^\star$ begins to decrease, after reaching its maximum of z_0, z remains constant* at $z_0-(f/r)$ until $z^\star$ has advanced (in the direction of motion) a distance f/r ahead of it, i.e. until $z^\star = z_0 - 2f/r$ (curve $B^\star C^\star$ of $z^\star$ and BC of z).

Starting from point C, z follows $z^\star$ again with a lag of f/r (in the direction of motion) until dz/dt changes sign (points $D^\star$ and D), and so on.

The curve $z = f(\omega t)$ is therefore made up of parts of the two sine curves $z = z_0 \sin \omega t \pm f/r$, joined by horizontal lines at $z = \pm[z_0-(f/r)]$.

It is difficult to define the phase displacement of the non-sinosoidal function z in relation to the sinusoidal function $z^\star$. We could perhaps use the first harmonic

* In effect, the applied force becomes less than the friction force. As long as z remains constant, dz/dt is zero, i.e. $|\sigma|$ varies from $+1$ to -1 between B and C.

of z. But an examination of *Figure 6.30* shows that we can get a good idea of it by evaluating the mean length of the lines $\alpha\beta$. For want of a better method, an approximate result can be obtained by calculating $A^{\star}/A'$ which is the shortest of the lines $\alpha\beta$

$$\phi = \widehat{A^{\star}A'} = \sin^{-1}\left(\frac{-f}{rz_0}\right)$$

This compares reasonably well with the value of

$$\tan^{-1}\left(-\frac{4}{\pi}\frac{f}{rz_0}\right)$$

obtained above.

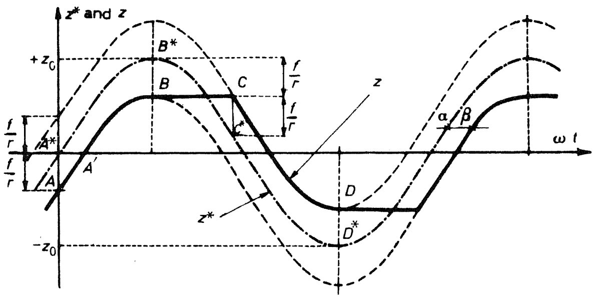


Figure 6.30

Research into the effect of backlash, j, on the motion of a transmission (position control) is made in the same way and with the same curve, using j instead of f/r (see the tracing of *Figure 7.42*, p. 312).

6.10.3. LINEAR SERVO SYSTEMS. THE VIBRATION DEAD ZONE

Consider a linear servo system, for example, a position control, consisting of a component B (mass m, stiffness r) whose input is the force F from component A. The input to component A is the error ϵ and $F/\epsilon = H$ (*Figure 6.31*).

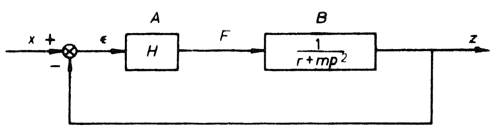


Figure 6.31

Such a system is unstable once component A exhibits the slightest phase lag, as can be seen in the Nyquist diagram (*Figure 6.32a*), and even more so if component A is an integrator (transfer function K/p), as shown in *Figure 6.32b* (this is the diagram for the hydraulic servo control to be described in Chapter 8).

Suppose that the damping coefficient, ζ_N to be applied to the transfer function of B in order to stabilize the control, is known and is small (this is usually the case in hydraulics).

The question arises whether a Coulomb friction force applied to B can stabilize the system. It was shown above that provided the friction force be small, it is equivalent to the damping coefficient of eqn. (7).
The condition for stability is $\zeta_0 > \zeta_N$, i.e.

$$\frac{2}{\pi}\frac{f}{rz_0}\frac{\omega_0}{\omega} > \zeta_n$$

or

$$z_0 < \frac{2}{\pi}\frac{f}{r\zeta_n}\frac{\omega_0}{\omega}$$

If $\omega \triangleq \omega_0$, this becomes

$$z_0 < \frac{2}{\pi}\frac{f}{r\zeta_n} \qquad (10)$$

Thus all oscillations of amplitudes less than

$$a_0 = \frac{2}{\pi}\frac{f}{r\zeta_n}$$

will be damped, but *if there is a large perturbation or a high-amplitude step input such that the system is moved from its equilibrium position by an amount greater than a_0, it will become unstable.*

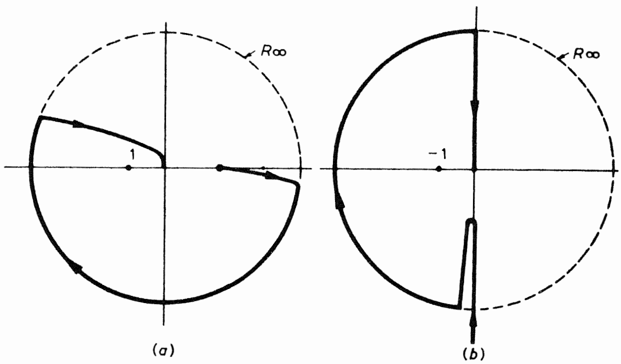


Figure 6.32

This value a_0, which is the minimum distance between the instantaneous position of the system B and its equilibrium position, is known as the *vibration dead zone.* Owing to the existence of this dead zone, we might well deduce that Coulomb friction can never stabilize a system with certainty, since a perturbation might develop which would cause a vibration amplitude greater than the dead zone. But in coming to this conclusion we have assumed that the system is strictly linear (except for the friction f), and this is never true. The analysis of the effect of Coulomb friction should therefore be made without the assumption of linearity. This will be done in the following Section.

Note—The vibration dead zone is well known to engineers who try to induce instability in the course of testing a servo system with Coulomb friction by using high-amplitude inputs.

The existence of a vibration dead zone is easy to establish. Consider an oscillation of amplitude z_0 in the absence of any input signal. Neither the restoring force, rz, nor the inertia force, $m(\mathrm{d}^2z/\mathrm{d}t^2)$, absorb or introduce energy to the system during a cycle.

It was shown in Section 6.10.1 that the friction force absorbed the energy

$$W_f = 4\,f\,z_0$$

This energy will be compared with the energy W_e introduced into the system by component A (the work done by the force F). We shall calculate the work done per cycle W_e.

It has been shown that if the friction force is small, we can write

$$z = z_0 \sin \omega t$$

Now. we have

$$F = -zH = -z_0 K(\omega) \cdot \sin[\omega t + \varphi(\omega)]$$

so that

$$W_e = \int_{\omega t=0}^{\omega t=2\pi} F\,\mathrm{d}z = -z_0^2 K(\omega) \int_{\omega t=0}^{\omega t=2\pi} \sin[\omega t + \varphi(\omega)] \cos \omega t \,\mathrm{d}(\omega t)$$

$$\boxed{W_e = -z_0^2 K(\omega)\pi \sin\varphi}$$

This gives the following results:

(1) *While the energy absorbed by Coulomb friction is proportional to the amplitude, the energy produced (or absorbed) by a linear element is proportional to the square of the amplitude.*

(2) For systems having a block diagram of the form shown in *Figure 6.31*, this latter energy is *positive* if the component A causes a *phase lag* ($\sin\phi$ negative in the expression for W_e).

(3) W_e is less than W_f at first but becomes equal to it when

$$a_0 = \frac{4\,f}{\pi K(\omega) \sin(-\varphi)}$$

where a_0 is the vibration dead zone.

Thus, for example, if A is a pure integrator (as in the servo control of Chapter 8) with a transfer function c/p,

$$K(\omega) = \frac{c}{\omega} \qquad \sin(-\varphi) = 1$$

so that

$$a_0 = \frac{4\,f\,\omega}{\pi c}$$

For a given value of $\omega = \omega_0$, W_f and W_e are represented on the W, z_0 diagram by a straight line passing through the origin and a parabola with its vertex at the origin, respectively. These curves intersect at a point which corresponds to the vibration dead zone. The curves are shown in *Figure 6.33*, together with a third which represents the energy supplied by a component which is initially linear (the part which is coincident with the parabola) but gradually becomes *saturated*. It can be seen that the vibration dead zone could disappear altogether for this type of component. This will be dealt with in more detail in the following Section.

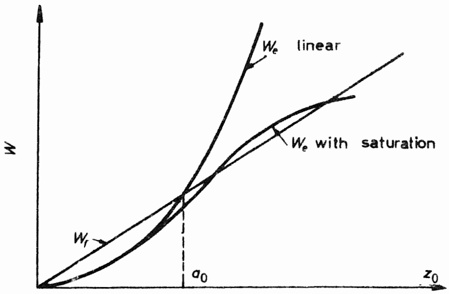


Figure 6.33

6.10.4. NON-LINEAR SERVO SYSTEMS

6.10.4.1. *Coulomb friction being the only source of damping*

Consider a servo system similar to the one referred to above but in which *the component A is not linear* (*Figure 6.34*).

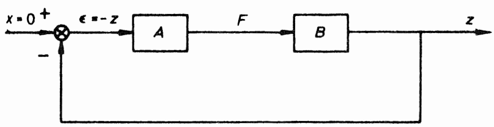


Figure 6.34

Let the equation of component B be

$$F = rz + f \cdot [\sigma] + m\frac{d^2z}{dt^2}$$

and let H be the transfer function of component A (a function of z_0, since A is not linear).

If $x = 0$, $\epsilon = -z$ and $F = -zH$, the equation for component B is

$$-zH(z) = rz + f \cdot [\sigma] + m\frac{d^2z}{dt^2}$$

This investigation will be confined to systems which are displaced from their equilibrium positions and then released, so that the resulting motion is practically sinusoidal (damped or amplified). This is true in any case if the Coulomb friction in component B is small and if the non-linear component A remains fairly regular (e.g. servo control valves which are non-linear due to dead zones or saturations).

We shall determine W_e, the work done per cycle by the force F, as a function of z_0, the amplitude of z. Since the work done by the restoring force and the inertia force is zero in a closed cycle (i.e. one which is neither amplified nor damped), the investigation of the stability of the system is reduced to a comparison of the work, W_e, done by the force F and the energy, W_f, absorbed by friction, the latter being $4fz_0$, as shown above. W_e and W_f can be plotted as a function of z_0 and the stability examined graphically, as shown in *Figure 6.33*.

Graphical representation on reduced coordinates—In practice, the plotting and graphical construction are made much easier if the coordinates (z_0, W) are replaced by (z_0, $W/4z_0$) where the friction energy is represented by a *horizontal* line with $W/4z_0 = f$; the energy introduced by a *linear component* is represented by a *straight* line passing through the origin (see p. 218, note 2).

$$\frac{W_e}{4z_0} = \frac{\pi}{4}K(\omega)z_0 \sin(-\phi)$$

and the energy introduced by a non-linear component is represented by a curve which is coincident with a straight line through the origin at first but which deviates from it as z_0 is increased (generally towards the base, see *Figure 6.35*). The curve $W/4z_0 = f(z_0)$ often has a maximum and takes the form shown in this figure.

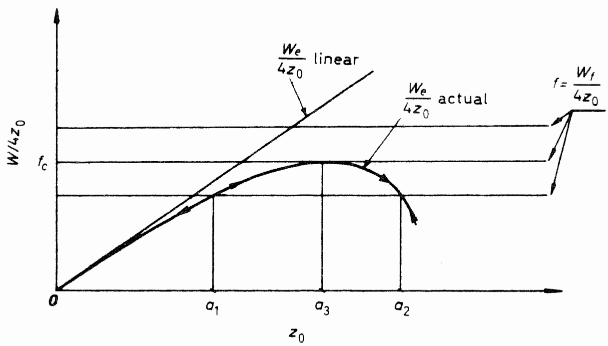


Figure 6.35

Thus, according to the value of f, there may be

(1) *two* amplitudes of continuous oscillation ($f < f_c$): a_1, which is unstable since $W_e - W_f$ increases with z_0, and a_2 which is stable since $W_e - W_f$ decreases with z_0;

(2) *one* amplitude of continuous unstable oscillation, a_3 (the double point for $f = f_c$);

(3) *no* amplitude of continuous oscillation ($f > f_c$). In this case the system will be stable for all perturbations applied.

The condition $f > f_c$ can be considered as the true criterion for stability. Figure 6.35 confirms that this condition cannot exist if we accept a linear approximation.

6.10.4.2. *Coulomb friction being insufficient to provide the necessary damping*

Very often the Coulomb friction is too small to provide sufficient damping. In this case the problem arises of estimating the supplementary artificial damping required. The solution of this problem is easy using the $(W/4z_0, z_0)$ diagram.

Suppose, for example, that the supplementary damping is linear; it absorbs energy W_s proportional to z_0^2 so that the curve of $W_s/4z_0$ against z_0 is a straight line passing through the origin. Stability will be obtained if the curve of

$$W_f/4z_0 + W_s/4z_0,$$

which represents the energy dissipated, always lies above the curve of $W_e/4z_0$, which represents the energy supplied.

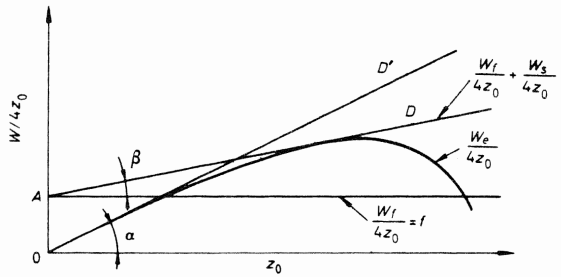


Figure 6.36

To determine the value of $W_s/4z_0$ strictly necessary for stability, all we have to do is to draw the tangent D to the curve $W_e/4z_0$ through point A ($z_0 = 0$, $W/4z_0 = f$). The slope, $\tan\beta$, of this line is then equal to $W_s/4z_0$ (see *Figure 6.36*).

In addition, we can often avoid the use of $W_s/4z_0^2$ in calculation of the necessary supplementary damping coefficient ζ_s'. Suppose that the necessary damping coefficient without friction is ζ_s, then ζ_s' and ζ_s are related by

$$\frac{\zeta_s'}{\zeta_s} = \frac{\tan\beta}{\tan\alpha}$$

where $\tan\beta$ is the slope of the tangent D to the curve of $W_e/4z_0$ from point A and $\tan\alpha$ is the slope of the curve of $W_e/4z_0$ at the origin (the straight line D' is the limit of D as f tends to zero).

In order to ensure a sufficient stability margin in the linear analysis of servo systems, it is necessary to increase the damping coefficient which is strictly required. We shall see below (Section 7.5.3) that a multiplication of the coefficient by a factor $\lambda = 2$ is necessary and sufficient to suppress any amplitude rise due to resonance in the open loop of servo controls with normal transfer functions.

The above method gives the artificial damping coefficient which is strictly necessary. The coefficient that is necessary in practice to ensure the required stability margin can be obtained if the construction is made on the curve $\lambda W_e/4z_0$ instead of the curve $W_e/4z_0$.

APPENDIX 6.1

COMPARISON OF THE PHASE TRAJECTORIES OF ACTUAL AND APPROXIMATE MOTIONS

The phase trajectory of a motion is the curve of $\mathrm{d}x/\mathrm{d}t$ against x. It may be found from the equation of motion and the initial conditions. We shall plot these curves for the two motions treated in Section 6.10.1.

Actual motion

The equation was

$$\frac{1}{\omega_0^2}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{f}{r}[\sigma] + x = 0$$

Putting

$$u = x + \frac{f}{r}[\sigma]$$

and
$$y = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} \quad \left(\text{except at } \frac{\mathrm{d}x}{\mathrm{d}t} = 0\right)$$

the equations of motion become

$$\left\{\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= -u\,\omega_0^2 \\ y &= \frac{\mathrm{d}u}{\mathrm{d}t} \end{aligned}\right\} \tag{1}$$

Multiplying these equations together term by term,

$$y\frac{\mathrm{d}y}{\mathrm{d}t} + u\,\omega_0^2\frac{\mathrm{d}u}{\mathrm{d}t} = 0, \tag{2}$$

integrating,

$$y^2 + u^2\omega_0^2 = C$$

and restating in terms of x gives

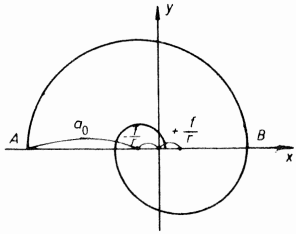
$$\boxed{\frac{y^2}{\omega_0^2} + \left[x + \frac{f}{r}[\sigma]\right]^2 = C} \tag{3}$$

where C is a constant of integration.

This equation is valid at all points except for $\mathrm{d}x/\mathrm{d}t = 0$; hence the constant of integration must be redetermined (using the condition of continuity) each time $\mathrm{d}x/\mathrm{d}t = 0$ $(y = 0)$.

The motion is represented on the x, y axes by a succession of semi-ellipses with a common axis $Ox(\mathrm{d}x/\mathrm{d}t = 0)$ and with successive centres at $\pm f/r$ (*Figure 6.37*). To make the curve easier to plot, we can make the substitution $Y = y/\omega_0$ so that it becomes a succession of semi-circles on the x, Y axes.

Figure 6.37



The work done against resistance in a half cycle, $W_r/2$, or more exactly, between the two maxima A and B must equal the difference in the potential energies for each maximum. Since these have abscissae of a_0+f/r and a_0-f/r, we have

$$\frac{W_r}{2} = \int_{a_0-f/r}^{a_0+f/r} r\,x\,\mathrm{d}x$$

$$= \left[\frac{r\,x^2}{2}\right]_{a_0-f/r}^{a_0+f/r} = \frac{r}{2}\left[\left(a_0 + \frac{f}{r}\right)^2 - \left(a_0 - \frac{f}{r}\right)^2\right]$$

$$\frac{W_r}{2} = 2\,f\,a_0$$

The *time taken to traverse the semi-circle* is given by

$$\frac{t}{2} = \int_A^B \mathrm{d}t = \int_A^B \frac{\mathrm{d}x}{\mathrm{d}x/\mathrm{d}t} = \int_A^B \frac{\mathrm{d}x}{y}$$

If we use the point $-f/r$ as the origin for the curve AB and choose the parameters

$$\begin{cases} u = \dfrac{a_0}{\omega_0} \cos\theta \\ y = a_0 \sin\theta \end{cases}$$

we have

$$\frac{t}{2} = \int_A^B \frac{\mathrm{d}x}{y} = \int_\pi^0 \frac{(-a_0/\omega_0)\sin\theta\,\mathrm{d}\theta}{a_0 \sin\theta} = \int_\pi^0 -\frac{\mathrm{d}\theta}{\omega_0} = \frac{\pi}{\omega_0}$$

This result shows that the introduction of Coulomb friction to an undamped second-order system does not change the angular frequency ω_0.

(The approximate motion has an angular frequency of $\omega_0\sqrt{1-\zeta_0^2}$ which is almost equal to ω_0 when ζ_0 is small.)

Approximate motion

The equation of motion is

$$\frac{1}{\omega_0^2}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{2\,\zeta_0}{\omega_0}\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0$$

so that, putting $y = dx/dt$, we have

$$\frac{dy}{dt} = -2\zeta_0\omega_0 y - \omega_0^2 x$$

The differential equation of the phase trajectory is therefore

$$\frac{dy}{dx} = -\omega_0^2 \frac{x}{y} - 2\zeta_0\omega_0$$

or

$$\frac{d(y/\omega_0)}{dx} = -\frac{x}{(y/\omega_0)} - 2\zeta_0$$

and by putting $Y = y/\omega_0$, as we did for the actual motion:

$$\frac{dY}{dx} = -\frac{x}{Y} - 2\zeta_0 \tag{4}$$

This equation can be integrated, and for the case in which we are interested, $0 < \zeta_0 < 1$, it becomes

$$x^2 + Y^2 + 2\zeta_0 xY = c\exp\left(\frac{2\zeta_0}{\sqrt{1-\zeta_0^2}}\tan^{-1}\frac{Y/x+\zeta_0}{\sqrt{1-\zeta_0^2}}\right) \tag{5}$$

Eqn. (4) can also be written in polar coordinates, and for
ρ = radius vector
θ = polar angle
ϕ = angle between the tangent to a point on the curve and the radius vector of the point
the equation of motion becomes

$$\frac{d\varrho/d\theta}{\varrho} = \tan\varphi = \zeta_0\frac{1-\cos 2\theta}{1+\zeta_0\sin 2\theta} \tag{4'}$$

Using eqn. (5) we can find the abscissae of two successive intersections with the Ox axis:

$$|x_1| = \sqrt{c}\exp\left(\frac{\zeta_0}{\sqrt{1-\zeta_0^2}}\psi\right) \quad \left(\text{where } \psi = \tan^{-1}\frac{\zeta_0}{\sqrt{1-\zeta_0^2}}\right)$$

$$|x_2| = \sqrt{c}\exp\left(\frac{\zeta_0}{\sqrt{1-\zeta_0^2}}(\psi-\pi)\right)$$

wherefrom (since ζ_0 is small)

$$\left|\frac{x_2}{x_1}\right| = \exp\left(-\frac{\zeta_0\pi}{\sqrt{1-\zeta_0^2}}\right) \simeq \exp(-\zeta_0\pi) \simeq 1-\zeta_0\pi$$

whereas for the actual motion

$$\left|\frac{x_2}{x_1}\right| = \frac{a_0 - f/r}{a_0 + f/r} \simeq 1 - \frac{2f}{ra_0}$$

so that, if the values of $|x_2/x_1|$ are put equal, the previous result is obtained again:

$$\zeta_0 = \frac{2}{\pi}\,\frac{f}{r\,a_0}$$

This equation also enables us to determine the ordinate at the intersection of the curve with the Oy axis which is situated between the two given intersections with Ox:

$$|\,Y\,| = \sqrt{c}\,\exp\left(-\frac{\zeta_0}{\sqrt{1-\zeta_0^2}}\,\frac{\pi}{2}\right)$$

wherefrom

$$\left|\frac{Y}{x_1}\right| = \exp\left[\frac{\zeta_0}{\sqrt{1-\zeta_0^2}}\left(\frac{\pi}{2}-\varphi\right)\right] \simeq \exp\left(-\zeta_0\frac{\pi}{2}\right) \simeq 1-\frac{\pi}{2}\zeta_0 \quad \left(=1-\frac{f}{r\,a_0}\right)$$

Thus

$$|\,Y\,| = \left|\frac{Y}{x_1}\right| x_1 \simeq \left(1-\frac{f}{r\,a_0}\right)\left(a_0+\frac{f}{r}\right) \simeq a_0$$

A certain amount of information which is not immediately evident in the integrated eqn. (5) can be obtained easily from the differential eqn. (4). In fact, from

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = -\frac{x}{Y} - 2\,\zeta_0$$

we can deduce that

(1) All the curves are geometrically similar, with a common centre at the origin. Multiplying x and Y by α does not change the differential equation. This is due to the linearity of the initial equation.

(2) The angle between the tangent to the curve at point x_0, Y_0 and the tangent to the circle passing through x_0, Y_0 with its centre at the origin, is always *less* than the small constant quantity $2\zeta_0$, i.e. at each point on the curve, in the direction corresponding to increasing time, the curve crosses to the inside of the circle passing through the point, centre O, in such a way that the angle between the tangents of the curve and the circle varies between $\tan^{-1} 2\zeta_0$ at $x = 0$ and 0 at $y = 0$.

(3) The line normal to the curve at any point, of

$$(x - x_0) - (y - y_0)\left(\frac{x_0}{Y_0} - 2\,\zeta_0\right) = 0$$

intersects the x axis at $x = -2\zeta_0 Y_0$, i.e. a point of intersection which varies between both sides of the point $x = -f/r$, the intercept of the normal to the actual phase trajectory with the x axis.

This consideration shows that the phase trajectory is a spiral. It is easier to plot by the graphic method of tangents (*Figure 6.38*) than by using the integrated equation, and in any case it is very similar to the succession of semi-circles which represent the phase trajectory of the actual motion.

Note—When the actual motion is damped, the succession of semi-circles will eventually intersect the x axis at a point $x = l$ lying between the points $x = \pm f/r$, so that it is impossible to plot the following semi-circle. Physically this means that movement ceases at $x = l$, since equilibrium takes place between the frictional force and the restoring force lr. This is obviously possible if $|lr| \leqslant f$ ($[\sigma] = lr/f$ is therefore understood to be between -1 and $+1$).

The actual motion no longer resembles the motion with viscous friction which in theory never stops.

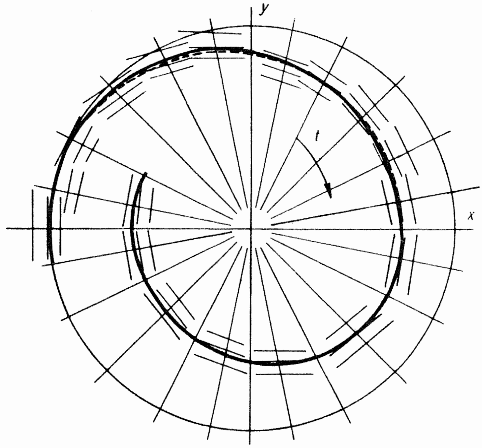


Figure 6.38. The dotted line is the phase trajectory of the actual motion with Coulomb friction (semi-circles)

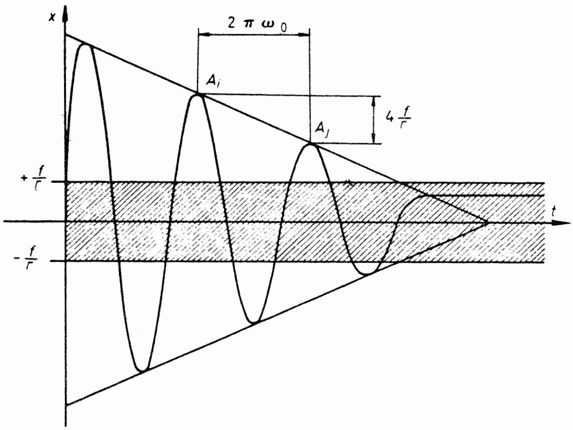


Figure 6.39

These factors enable us to distinguish easily between recordings of the motions with dry friction and with viscous friction. While the envelope of a damped sinusoidal oscillation is the exponential curve $\exp(-\zeta_0\omega_0 t)$, for the motion with Coulomb friction the locus of the successive maxima of the $x(t)$ curve consists of two converging straight lines (*Figure 6.39*), since if we consider any two successive maxima of the same sign, A_i and A_j, their difference in the abscissa equals the periodic time—a constant, and that in the ordinates equals $4f/r$—also a constant.

In practice, most recordings show that the envelope of a damped sine curve, which is at first a concave exponential, ends in straight lines tangential to the exponential curve, so that below a certain amplitude the Coulomb friction effect predominates over the viscous friction.

APPENDIX 6.2

HALL AND NICHOLS CHARTS

Hall and Nichols charts enable us

(1) to make the complex transformation $Z = z/1+z$ graphically;
(2) to investigate the stability of a closed-loop system with unit feedback $H/1+H$.

The Hall chart (Chapter 11, Graph I)

$z = \rho e^{i\theta}$ is plotted in polar coordinates.

The value of $Z = Pe^{i\Theta}$ may be read directly from the chart, interpolating P and Θ from curvilinear coordinates formed from the circles P = constant and Θ = constant.

The Nichols chart (Chapter 11, Graph J)

$z = \rho e^{i\theta}$ is plotted in semi-logarithmic coordinates (abscissae θ, ordinates $A = 20 \log_{10}\rho$). The corresponding value of $Z = Pe^{i\Theta}$ is read directly from the curves of P = constant and Θ = constant.

Note—In the Hall chart, the curves of constant P and constant Θ are circles. The complex member $Z = z/1+z$ has the cartesian coordinates

$$X = \frac{x^2 + y^2 + x}{(x+1)^2 + y^2}$$

$$Y = \frac{y}{(x+1)^2 + y^2}$$

Curves of constant Θ are therefore given by

$$(x + \tfrac{1}{2})^2 + \left(y - \frac{1}{2\tan\Theta}\right)^2 = \left(\frac{1}{2\sin\Theta}\right)^2$$

We also have

$$(X-1)^2 + Y^2 = P^2 + 1 - 2X = \frac{1}{(x+1)^2 + y^2}$$

so that curves of constant P are given by

$$x^2 + y^2 + 2x\frac{P^2}{P^2-1} + \frac{P^2}{P^2-1} = 0$$

These are circles of centre $x = -P^2/(P^2-1)$, $y = 0$ which intersect the y axis at $x_1 = -P/(P-1)$ and $x_2 = -P/(P+1)$.

THE HYDRAULIC SERVOCONTROL

7.1. OUTLINE

Hydraulic servocontrols are defined as all *servocontrol systems with hydraulic power amplification*. They are chiefly met with in machine tools (control of movements, reproducing systems, etc.), in aircraft and missiles (control of ailerons, rudder, etc.), in vehicles (control of direction) and in gun turrets.

There are many different types of servocontrols:

the input can be a position, a force or of a different nature (electrical or hydraulic);

the output can be a position but also a speed or even a force;

the amplification may be carried out by a linear motor (single- or double-acting ram) or a rotating motor (positive displacement or otherwise);

the feedback chain can be kinematic (mechanical servocontrol), electrical (feedback potentiometer), hydraulic, etc.

However, all hydraulic servocontrols have transfer functions of the same type and in particular they all have *inadequate natural damping*.

This close relationship between the different types of servocontrols enables us to break down the general analysis into (*a*) a detailed analysis of a particular type; (*b*) a brief review of properties peculiar to some of the main types.

The servocontrol chosen for detailed analysis is the simplest: the so-called mechanical servocontrol with an equal-area ram and having *a position gain of* 1. The analysis will include the description; the derivation of equations for determining the size of the equipment; definition of characteristic coefficients; estimation and analysis of the transfer function; a brief examination of the devices available for damping; and an account of the effect of certain external parameters, such as the rigidity of attachments and linkages.

The second part will deal briefly with servocontrols of the following types:

mechanical with gain $\neq 1$
mechanical with single-acting ram
mechanical with rotational motor
electro-hydraulic (electrical input and feedback)
and, finally, the servocontrol whose output is a force instead of a position.

Although the results obtained are of general validity, the numerical examples quoted have been taken from the field of aeronautical engineering.

7.2. DESCRIPTION OF THE 'MECHANICAL' SERVOCONTROL

A symmetrical equal-area ram has one part fixed to the structure at F and the other part to the control lever at N. The former can be either the body of the ram (*Figure 7.1a*) or the piston (*Figure 7.1b*).

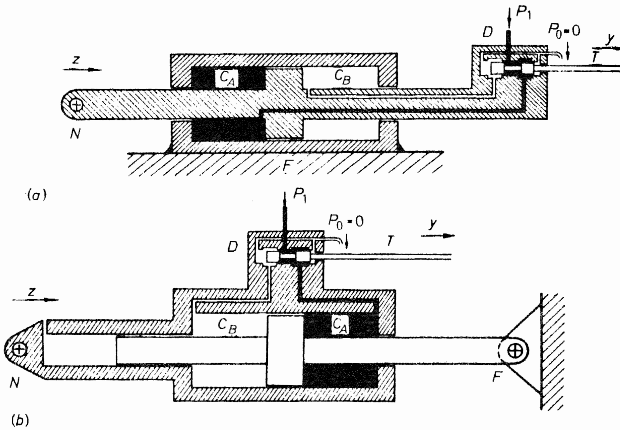


Figure 7.1

The two chambers, C_A and C_B, of the ram are connected, one to a source of high pressure, P_1, the other to the sump P_0, through a control valve, D, whose piston is attached to the input linkage rod, T. *The movement of the ram is therefore controlled by the relative opening of the control valve.*

Let

y = absolute position of the valve piston
z = position of point N (output)
e = relative opening of the valve ($e = y - z$).

e is the error, and the transfer function, $H_0 = z/e$, is the open-loop transfer function of the servocontrol, whose block diagram is given in *Figure 7.2*.

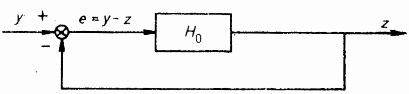


Figure 7.2

It is important to note that it is the fixing of the *body* of the control valve to the *movable* part of the ram which *kinematically* creates the feedback or, more precisely, unit feedback.

The input linkage is often attached to the control valve by means of a small lever arm fixed to the movable part of the ram.

If x is the movement of the control attachment point and λ the multiplication factor of the linkage ($\lambda = \alpha\gamma/\beta\gamma$), it can be seen (*Figure 7.3*) that

$$e = y - z = \frac{1}{\lambda}(x - z) = \frac{x}{\lambda} - \frac{z}{\lambda}$$

The linkage has effectively divided both the input x and the feedback signal z by λ. The real block diagram is that shown in *Figure 7.3c*, but it is more convenient to give it in the equivalent form shown in *Figure 7.3d*.

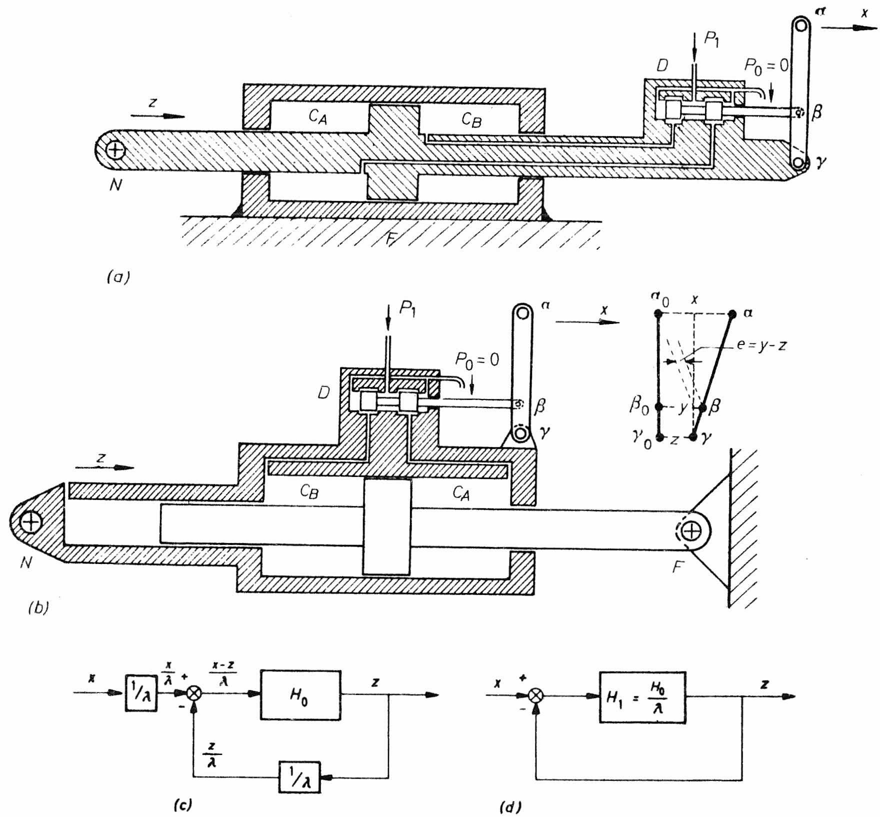


Figure 7.3

This servocontrol linkage will be investigated in the following Sections. We will assume the layout shown in *Figure 7.3b* but the results could equally well apply to that given in *Figure 7.3a*.

The results for the servocontrol with direct control attachment shown in *Figure 7.1* can obviously be obtained by putting $\lambda = 1$ in the equations derived for *Figure 7.3b*.

7.3. ESTIMATION OF DIMENSIONS. CHARACTERISTIC COEFFICIENTS

While analysing the performance of a given servocontrol, we shall at the same time determine the size and dimensions of the servocontrol necessary to comply with certain performance requirements. In order to do both of these things from one analysis, it is necessary to refer constantly to the basic equations relating performance and dimensions, and it is useful to define certain dimensionless coefficients.

A servocontrol can also be directly attached to a part moving in *translation* or, by means of a lever arm, to a part moving in *rotation* so that, for each equation established for an output in translation, there will be a corresponding one for a rotational output, provided that the angular movement is small enough for the inclination of the lever arm to be neglected.

Finally, we must note that in all these equations the back pressure, P_0, is assumed to be zero. If this is not the case, P_1 must be replaced by $(P_1 - P_0)$ wherever it occurs.

7.3.1. PRINCIPAL NOTATION

P_1 = hydraulic supply pressure
S = effective area of the ram (annular surface)
z_M = maximum half travel of the ram (amplitude of z)
l* = length of lever arm
β* = angle of rotation ($\beta = z/l$)
F_M = maximum force applied by the ram ($F_M = P_1 S$) (neglecting friction in the ram)
C_M* = maximum couple provided by the ram ($C_M = P_1 Sl$)
F_r = resisting force applied to the ram (maximum value F_R)
C* = resisting torque applied to the ram (maximum value C_R)
V_e = effective half volume ($V_e = Sz_M = Sl\beta_M$)
V_t = total half volume of the ram, i.e. the volume provided for the liquid downstream of the valve in one chamber or the other when the ram is in the middle position

$(\mathrm{d}z/\mathrm{d}t)_M$ = maximum linear velocity
$(\mathrm{d}\beta/\mathrm{d}t)_M$* = maximum angular velocity
Q = hydraulic volume flow

7.3.2. THE POWER COEFFICIENT (OR COEFFICIENT OF EXCESS POWER), k_s

$$k_s = \frac{F_M}{F_R} = \frac{P_1 S}{F_R} \quad \text{or} \quad k_s = \frac{C_M}{C_R} = \frac{P_1 S l}{C_R}$$

k_s is the ratio of the maximum force available to the maximum resisting force. It is normally in the neighbourhood of 1. It must be increased, however,

(1) if the dynamic performance of the servocontrol is to be almost unaffected by the *load*, F_r.

F_r can very often be considered as proportional to z. For a ram with k_s in the neighbourhood of 1, as z approaches z_M a larger and larger part of P_1 is used to overcome F_r and the loss of head available for the valve decreases, so that the movement of the ram slows down, with the result that the dynamic performance deteriorates.

* *For rotational output*

(2) if the required performance results in inertia forces which are large compared with the restoring forces (e.g. the servocontrol for a radar aerial).

7.3.3. THE VOLUMETRIC COEFFICIENT, k_v

$$k_v = \frac{V_t}{V_e} = \frac{V_t}{S z_M} \quad \text{or} \quad \frac{V_t}{S l \beta_M}$$

It is always desirable to decrease the total volume, V_t. In a *linear* actuator, $V_t = V_e + V_m$, where V_m is the dead volume corresponding to the ducting between the distributor and the actuator and to other intermediate volumes.

We can say that k_v is a measure of the *quality of the design*. For large rams, k_v can be as low as 1·2, while for small rams it is rarely less than 1·5 or 1·6.

It will be shown below (Section 7.9.3) that it is possible to obtain values of the coefficient k_v less than 1 with rotational hydraulic motors.

As will be seen, the use of the coefficients k_s and k_v facilitates the estimation of the performance limits during the initial design stages.

7.3.4. EQUATIONS RELATING TRANSLATIONAL AND ROTATIONAL OUTPUTS

The two fundamental equations are

$$\begin{cases} r z + f \dfrac{dz}{dt} + m \dfrac{d^2 z}{dt^2} = F = \Delta P\, S \\ c \beta + \psi \dfrac{d\beta}{dt} + I \dfrac{d^2 \beta}{dt^2} = C = \Delta P\, S l \end{cases}$$

where

r = restoring stiffness
f = coefficient of viscous friction
m = mass
c = rotational restoring stiffness
ψ = the rotational coefficient of viscous friction
I = moment of inertia
ΔP = difference between the pressures in the two chambers of the ram

Replacing z by βl in the first equation and multiplying through by l gives

$$r l^2 \beta + f l^2 \frac{d\beta}{dt} + m l^2 \frac{d^2\beta}{dt^2} = \Delta P\, S l$$

so that, by comparison with the second equation, we have the relationships

$$\begin{cases} \mathbf{c} = \mathbf{r l^2} \\ \boldsymbol{\psi} = \mathbf{f l^2} \\ \mathbf{I} = \mathbf{m l^2} \end{cases}$$

which, together with

$$\begin{cases} \boldsymbol{\beta} = \dfrac{\mathbf{z}}{\mathbf{l}} \\ \mathbf{C} = \mathbf{F l} \end{cases}$$

Table 7.1 Chief equations for determination of dimensions

	Translational output	*Rotational output*
Maximum force from the ram or torque, neglecting friction in the ram	$F_M = P_1 S$	$C_M = P_1 S l$
Effective area necessary (or product Sl)	$S = \dfrac{F_M}{P_1} = \dfrac{k_s F_R}{P_1}$	$S l = \dfrac{C_M}{P_1} = \dfrac{k_s C_R}{P_1}$
Maximum flow necessary	$Q_M = S\left(\dfrac{dz}{dt}\right)_M = \dfrac{k_s F_R}{P_1}\left(\dfrac{dz}{dt}\right)_M$	$Q_M = Sl\left(\dfrac{d\beta}{dt}\right)_M = \dfrac{k_s C_R}{P_1}\left(\dfrac{d\beta}{dt}\right)_M$
Maximum hydraulic power necessary, $H_M = P_1 Q_M$	$H_M = k_s F_R \left(\dfrac{dz}{dt}\right)_M$	$H_M = k_s C_R \left(\dfrac{d\beta}{dt}\right)_M$
Effective half volume	$V_e = S z_M = \dfrac{k_s F_R}{P_1} z_M$	$V_e = S l \beta_M = \dfrac{k_s C_R}{P_1} \beta_M$
Total half volume $V_t = k_v V_e$	$V_t = \dfrac{k_s k_v F_R}{P_1} z_M$	$V_t = \dfrac{k_s k_v C_R}{P_1} \beta_M$
Mean hydraulic flow necessary to give N c/s of amplitude z_M (or β_M), $Q_M = 4NV_e$	$Q_m = \dfrac{4 N k_s F_R z_M}{P_1}$	$Q_m = \dfrac{4 N k_s C_R \beta_M}{P_1}$
Maximum instantaneous flow necessary in the same conditions (the cycles being sinusoidal), $Q_i = \dfrac{\pi}{2} Q_m$	$Q_i = \dfrac{2 \pi N k_s F_R z_M}{P_1}$	$Q_i = \dfrac{2 \pi N k_s C_R \beta_M}{P_1}$

allow us to convert an equation for linear motion (variable z) to one for rotational motion (variable β) and vice versa.

7.4. DERIVING THE TRANSFER FUNCTION EQUATION (ASSUMING LINEARITY)

7.4.1. ASSUMPTIONS

The equation derived below applies to a given servocontrol with a given supply pressure, P_1, and attached to a given load. It is assumed that

the load consists of a restoring force proportional to z ($F_r = rz$), a viscous frictional force and an inertia force;

the variations of z are of small amplitude, with the middle position as the origin (so that the total volumes, V_t, of the two chambers are effectively equal);

the control valve is symmetrical, linear and has no dead zone or overlap, i.e. the pressures in the two chambers are equal to $P_1/2$ when there is no error signal and remain in the neighbourhood of $P_1/2$ during movements up to fairly high frequencies. Under these conditions, i.e. for a loss of head of $P_1/2$, the gains of each half of the control valve (relative gains of each chamber: $K_A = Q_A/e$, $K_B = Q_B/e$) are equal, opposite and *constant;*

there is no backlash or elasticity at the connecting linkages and attachment points.

Despite these restricting conditions, the resulting analysis is extremely important, since

(*a*) the main problems of servocontrols, i.e. those of *stability* and *accuracy*, conform to the assumption of small movements;

(*b*) the transfer functions remain *practically* constant for movements about a position other than the middle, for movements of significant amplitude and for forces which are not linear in z, provided that k_s is not too close to unity. (*Figure 7.4* gives the natural frequency of the servocontrol (defined in Section 7.5.2) as a function of its mean position.)

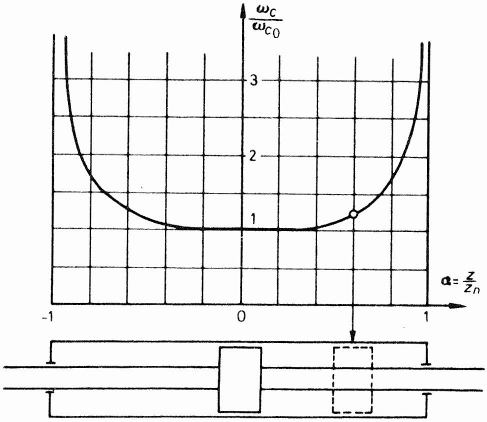


Figure 7.4. Theoretical variation of ω_c/ω_{c0} with z/z_M

ω_{c2}: critical frequency about $z = 0$. $\omega_{02} = \sqrt{2BS^2/V_t m}$
ω_c: critical frequency about $z \neq 0$

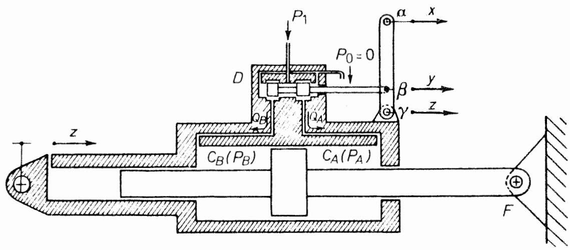
This analysis is the basis of a slightly more complex one, which will be made later, to take into account non-viscous friction, non-linearity of the control valve and elasticity of the attachment and fixing points (in particular, Section 7.6.2.1 includes the derivation of an equation for a valve without the linear assumption of constant loss of head).

7.4.2. DETERMINATION OF THE TRANSFER FUNCTION

The positive directions are taken as shown in *Figure 7.5* and the following notation is used:

Q_A, Q_B = flows *into* chambers C_A and C_B
P_A, P_B = pressures *inside* these chambers
B = bulk modulus of the oil (cf. p. 145).

Figure 7.5



Kinematic equation

$$x - z = \frac{\alpha\gamma}{\beta\gamma}(y - z) = \lambda(y - z) = \lambda e \tag{1}$$

Flow equations

$$Q_A = K e = \frac{K}{\lambda}(x - z) = S\frac{dz}{dt} + \frac{V_t}{B}\frac{dP_A}{dt} \tag{2}$$

$$Q_B = -K e = \frac{-K}{\lambda}(x - z) = -S\frac{dz}{dt} + \frac{V_t}{B}\frac{dP_B}{dt} \tag{3}$$

Force equation

$$(P_A - P_B)S = rz + f\frac{dz}{dt} + m\frac{d^2z}{dt^2} \tag{4}$$

(2) − (3) gives

$$\frac{K}{\lambda}(x - z) = S\frac{dz}{dt} + \frac{V_t}{2B}\left(\frac{dP_A}{dt} - \frac{dP_B}{dt}\right) \tag{5}$$

wherefrom, using the Laplace transform

$$P_A - P_B = z\frac{1}{S}[r + fp + mp^2] \tag{4'}$$

$$\frac{K}{\lambda}(x - z) = zpS + \frac{V_t}{2B}p(P_A - P_B) \tag{5'}$$

and, by eliminating $P_A - P_B$

$$z\left[pS + \frac{V_t}{2BS}p(r + fp + mp^2)\right] = \frac{K}{\lambda}(x - z)$$

which gives the required transfer function

$$\boxed{H_1 = \frac{z}{x - z} = \frac{K/\lambda S}{p\left[1 + \frac{V_t r}{2BS^2} + \frac{V_t f}{2BS^2}p + \frac{V_t m}{2BS^2}p^2\right]}} \tag{6}$$

If the output is rotational, this becomes:

$$H_1' = \frac{\beta l}{x - \beta l} = \frac{K/\lambda S}{p\left[1 + \frac{V_t c}{2B(Sl)^2} + \frac{V_t \varphi}{2B(Sl)^2}p + \frac{V_t I}{2B(Sl)^2}p^2\right]} \tag{6'}$$

Note that (2) + (3) gives

$$\frac{V_t}{B} \cdot \frac{\mathrm{d}(P_A + P_B)}{\mathrm{d}t} = 0$$

$\therefore \quad P_A + P_B = \text{constant}$

Since initially $P_A = P_B = P_1/2$, we have

$$P_A = \frac{P_1}{2} + \frac{P_A - P_B}{2}$$

$$P_B = \frac{P_1}{2} - \frac{P_A - P_B}{2}$$

i.e. P_A and P_B are 'symmetrical' with respect to $P_1/2$.

7.5. ANALYSIS OF THE SERVOCONTROL (ASSUMING LINEARITY)

7.5.1. PRELIMINARY ANALYSIS, WITH CONTROL VALVE CLOSED

Consider the servocontrol *filled with oil*, attached to the load and suppose that the orifices are closed.

The assembly constitutes a mechanical system of the second order, with mass, m, coefficient of viscous friction, f, and restoring stiffness, $r + r_h$, where r_h is the hydraulic stiffness of the oil in the ram.

Estimation of r_h—A displacement, Δz, of the ram gives an absolute variation in the volume of the two chambers of the ram

$$\Delta V = \pm \Delta z\, S$$

By definition of the bulk modulus, this variation gives a change in pressure

$$\Delta P = \pm B \frac{\Delta V}{V_t} = \pm \frac{B \Delta z S}{V_t}$$

which produces a restoring force

$$\Delta F = 2 \Delta P S = \frac{2 B S^2}{V_t} \Delta z$$

Hence, the corresponding stiffness is

$$r_h = \frac{\Delta F}{\Delta z} = \frac{2 B S^2}{V_t}$$

The transfer function [position/external force] of this system will therefore be

$$H_f = \frac{z}{\varphi} = \frac{1}{r + \dfrac{2 B S^2}{V_t} + f p + m p^2} = \frac{V_t/2 B S^2}{1 + \dfrac{V_t r}{2 B S^2} + \dfrac{V_t f}{2 B S^2} p + \dfrac{V_t m}{2 B S^2} p^2}$$

With the exception of the integration factor, this transfer function has the same denominator as that of the servocontrol given above.

If we now consider the servocontrol with the ram clamped and the control valve open, we can readily find the transfer function, $H_b = \phi/(x-z)$—the force exerted on the ram divided by the error.

The flow from the control valve is equal to $Q = Ke = K(x-z)/\lambda$, so that the change of 'volume' in each chamber is

$$\Delta V = \int Q \, \mathrm{d}t = \frac{Q}{p} = \frac{K(x-z)}{\lambda} \frac{1}{p}$$

Since the ram is fixed, this is exactly counterbalanced by a change in pressure

$$\Delta P = B \frac{\Delta V}{V_t} = \frac{B}{V_t} \frac{K(x-z)}{\lambda} \frac{1}{p}$$

which gives a force:

$$\varphi = 2 \Delta P S = \frac{2 B S}{V_t} \frac{K(x-z)}{\lambda} \frac{1}{p}$$

so that $$H_b = \frac{\varphi}{x-z} = \frac{2 B S}{V_t} \frac{K}{\lambda} \frac{1}{p}$$

The transfer function, H_1, of the servocontrol is equal to the product, $H_f . H_b$, of the transfer function with the control valve closed and that with the ram clamped.

7.5.2. SIMPLIFIED TRANSFER FUNCTION

The hydraulic stiffness, r_h, is always very much greater than the restoring stiffness, r. The term

$$\frac{V_t r}{2 B S^2} = \frac{r}{r_h}$$

is therefore *small compared with 1*; r/r_h can be found using the values of V_t and S given in *Table 7.1*:

$$r_h = \frac{2\,B\,S^2}{V_t} = 2\,B\,\frac{k_s^2\,F_R^2}{P_1^2}\;\frac{P_1}{k_s\,k_v\,F_R\,z_M} = \frac{2\,B}{P_1}\;\frac{k_s}{k_v}\;\frac{F_R}{z_M}$$

$$\frac{r}{r_h} = \frac{P_1}{2\,B}\;\frac{k_v}{k_s}\;\frac{r}{F_R/z_M}$$

If the ram is well designed, k_v will only just exceed unity and in any case will not be greater than 2; k_s is never less than unity, so that k_v/k_s must be less than 2.

The highest supply pressures never actually exceed 300 kg/cm^2 and the lowest values of the bulk modulus are about 10,000 kg/cm^2. $P_1/2B$ will therefore be less than $\frac{1}{100}$.

Finally, if F_r is a linear function of z, $F_R/z_M = r$, then r/r_h will be less than 0·02 and can be neglected compared with 1. In the following analysis we will use the simplified form of the transfer function

$$H_1 = \frac{z}{x - z} = \frac{K/\lambda\,S}{p\left[1 + \dfrac{V_t f}{2\,B\,S^2}\,p + \dfrac{V_t m}{2\,B\,S^2}\,p^2\right]} \tag{7}$$

which is generally written in the canonic form

$$\boxed{H_1 = \frac{z}{x - z} = \frac{\omega_f}{p\left[1 + 2\,\zeta\,\dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right]}} \tag{8}$$

where

$$\omega_f = \frac{K}{\lambda\,S} \tag{9}$$

$$\omega_c = \sqrt{\frac{2\,B\,S^2}{V_t\,m}} = \sqrt{\frac{r_h}{m}} \tag{10}$$

$$\zeta = \frac{\omega_c}{2}\;\frac{V_t f}{2\,B\,S^2} = \frac{\omega_c}{2}\;\frac{f}{r_h} = \frac{1}{2}\,f\sqrt{\frac{1}{m\,r_h}} \tag{11}$$

7.5.3. ANALYSIS OF THE SERVOCONTROL USING THE SIMPLIFIED TRANSFER FUNCTION

The use of reduced variables enables us to represent the transfer loci of all the servocontrols for which the linear assumption is valid by a single family of curves (cf. Section 7.4.1).

Figure 7.6 gives the curves

$$A = f\left(\frac{\omega}{\omega_c}\right) = \frac{1}{\mathrm{j}\dfrac{\omega}{\omega_c}\left[1 + 2\mathrm{j}\zeta\dfrac{\omega}{\omega_c} - \left(\dfrac{\omega}{\omega_c}\right)^2\right]}$$

for different values of ζ (for a more detailed plot, see Graph E). Using these curves we can deduce the loci with which we are concerned

$$\frac{\omega_f/\omega_c}{\mathrm{j}\dfrac{\omega}{\omega_c}\left[1 + 2\mathrm{j}\zeta\dfrac{\omega}{\omega_c} - \left(\dfrac{\omega}{\omega_c}\right)^2\right]}$$

by vertical translation through $+20\log(\omega_f/\omega_c)$.

The behaviour of the servocontrol (assuming linearity) can thus be determined completely from *Figure 7.6*. The more important results will be discussed below.

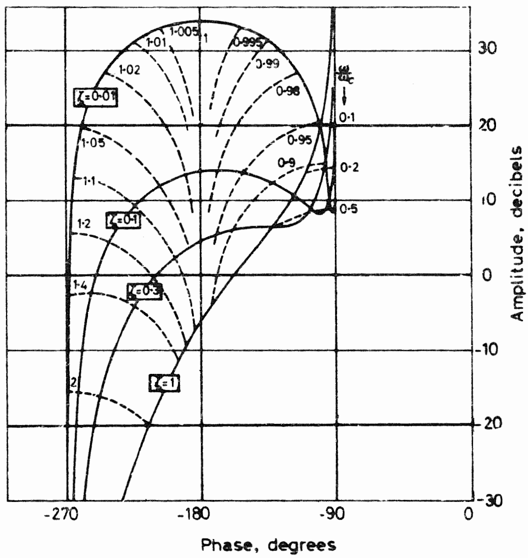


Figure 7.6. Curves of $F(jw) = \dfrac{1}{j\dfrac{\omega}{\omega_c}\left[1+2\zeta_f\dfrac{\omega}{\omega_c}+\left(j\dfrac{\omega}{\omega_c}\right)^2\right]}$ for different values of ζ (for a more detailed plot, see Chapter 11, Graph *E*)

The significance of ω_c *(critical frequency)*—The angular frequency, $\omega = \omega_c$, merits special attention. It represents the frequency at which the phase difference of the open loop, H_1, reaches 180°.

If the gain is increased until instability appears, ω_c will correspond to the resonant frequency.

On the other hand, the phase difference of the closed loop is also 180° at $\omega = \omega_c$.

ω_c should therefore always be greater than the frequency used, for which the maximum phase difference normally allowed is of the order of tens of degrees: *the value of ω_c should be as large as possible.*

For an existing ram—The equation derived above

$$\omega_c = \sqrt{\frac{2\,B\,S^2}{V_t\,m}} \tag{10}$$

shows that for an existing ram (S and V_t given) the increase in ω_c can be achieved only by decreasing the mass of the load, m, or by increasing the bulk modulus, B, both of which are seldom possible in practice.

For a proposed ram design—Transforming the expression for ω_c, using *Table 7.1*, we replace S by $k_s F_R/P_1$ (or Sl by $k_s C_R/P_1$) and V_t by $k_s k_v F_R z_M/P_1$ (or by $k_s k_v C_R \beta_M/P_1$), and we have

$$\omega_c = \sqrt{\frac{2\,B}{P_1}\,\frac{k_s}{k_v}\,\frac{F_R}{m z_M}} \quad \left(\text{or } \omega_c = \sqrt{\frac{2\,B}{P_1}\,\frac{k_s}{k_v}\,\frac{C_R}{I\beta_M}}\right) \tag{11}$$

F_R, m and z_M (or C_R, I and β_M) are usually determined by other considerations, as is the choice of liquid (so that B is fixed)*.

The increase in ω_c can only be achieved by

decreasing P_1; this favourable effect is due to the form of the relationship defining B

$$\frac{\mathrm{d}V}{V} = -\frac{1}{B}\,\mathrm{d}P$$

and does not exist for gases, for which

$$\frac{\mathrm{d}V}{V} = -\gamma\frac{\mathrm{d}P}{P}$$

(see Section 5.3.2);

decreasing k_v (this is obvious: a decrease in k_v indicates a decrease in the total volume as a result of improving the design);

increasing k_s (this is self-explanatory without analysis, if we realize that, for example, doubling the effective area of the ram is the same as attaching two half masses to two of the original rams).

Unfortunately, a decrease in P_1 is seldom possible, as it often leads to a prohibitively large increase in the overall space occupied by the equipment (and to a wastage of energy if, for example, the pressure has to be lowered in equipment where the source is common to several circuits and its pressure cannot be changed); a decrease in k_v reaches its limit; an increase in k_s entails an increase in weight, bulk and energy consumption. *To sum up, the critical*

* Note that although F_R and m (or C_R and I) are generally carefully examined, less attention is paid to z_M (or β_M); ω_c can be increased by reducing z_M or β_M just as easily as by reducing m or I.

frequency, ω_c, which is difficult to change, represents a limitation of the performance of servocontrols.

Note, however, that if the ram is replaced by a rotational motor, values of k_v less than unity can be attained so that ω_c is increased. The decrease in k_v is achieved by reducing the size of the motor while increasing its speed of rotation (reduction ratio). Theoretically there is no lower limit to k_v. In practice, there is a limit to the decrease in size of the motor. On the other hand, as the speed of rotation increases, the inertia forces increase and reach a stage when they are no longer negligible.

The significance of ω_f— $\omega_f = K/\lambda S$ is a measure of the open-loop gain, H_1. The closed-loop transfer function is

$$H = \frac{H_1}{1 + H_1} = \frac{z}{x} = \frac{1}{1 + \dfrac{p}{\omega_f}\left(1 + 2\zeta\dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right)} \tag{12}$$

If, as is often the case, ω_f is much less than ω_c and ζ is fairly small, the transfer function can be represented by the first-order function H', at least up to $\omega = \omega_f$:

$$H' = \frac{1}{1 + p/\omega_f}$$

ω_f is the reciprocal of the time constant of the approximate transfer function and is therefore fairly approximately equal to the angular frequency at which the phase difference in the open loop is 45°.

Since the frequency at which the phase difference is 45° is very often specified by the user, the manufacturer has even less control of ω_f than of ω_c.

Stability—The *absolute* condition for stability is

$$|H_1| < 1 \quad \text{for} \quad \omega = \omega_c$$

For $\omega = \omega_c$, H_1 reduces to

$$H_{1(\omega_c)} = \frac{\omega_f}{j\omega_c\left(2\zeta\dfrac{j\omega_c}{\omega_c}\right)} = \frac{-\omega_f}{2\zeta\omega_c}$$

The condition becomes

$$\boxed{\zeta > \frac{1}{2}\frac{\omega_f}{\omega_c}} \tag{13}$$

or, substituting for ζ and ω_c

$$\boxed{f > m\,\omega_f}$$
$$\boxed{(\psi > I\,\omega_f)} \tag{14}$$

The absolute condition of stability is clearly insufficient. Using Graph E the necessary value of ζ for each value of ω_f/ω_c has been determined, such that (1) there is no amplitude increase due to resonance in the closed loop; (2) this increase does not exceed 2·3 dB.

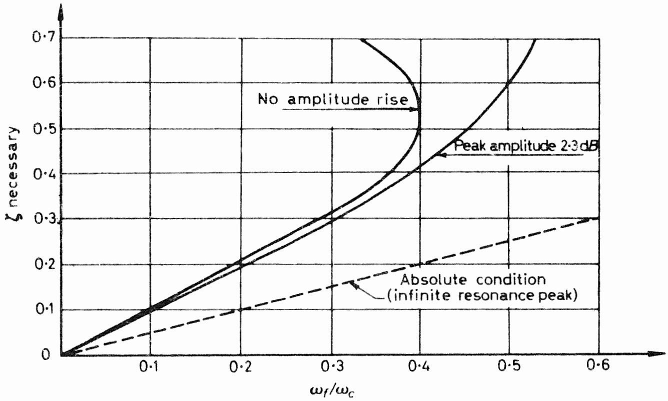


Figure 7.7

The results are shown in *Figure 7.7*. From this figure we can see that

(*a*) for $\omega_f/\omega_c > 0{\cdot}5$, stability is almost impossible to achieve;

(*b*) for $\omega_f/\omega_c < 0{\cdot}35$, the condition for stability can be represented fairly accurately by the relationship

$$\boxed{\zeta > \frac{\omega_f}{\omega_c}} \tag{15}$$

which becomes

$$\boxed{f > 2\,m\,\omega_f} \qquad \boxed{(\psi > 2\,I\,\omega_f)} \tag{16}$$

This is the absolute condition of stability within a factor of about 2.

These equations are extremely important. In practice, the frictional force, f (or ψ), of a servocontrol is generally very small, so that ζ is also very small; the critical frequency, ω_c, cannot be altered; the value of ω_f is often specified.

The problem to be faced in the design of a servocontrol is therefore: in the case where the servocontrol is practically possible, i.e. if ω_f/ω_c is less than about 0·4, to provide a friction f (or ψ) or even better, to introduce by indirect means a damping coefficient ζ such that eqn. (15) is satisfied. There are many ways of doing this, but they all have their cost in terms of *accuracy, energy or complexity*.

Some of them cannot be investigated using linear methods. We will therefore divert our attention to the non-linear analysis of servocontrols before considering them.

7.6. NON-LINEAR STABILITY ANALYSIS

7.6.1. INADEQUACY OF THE LINEAR METHOD; BASIS OF THE NON-LINEAR METHOD

When the linear method predicts instability in a servocontrol, this is usually confirmed in practice. But the experimental oscillations are limited to small amplitudes which are practically constant, while the linear method could not predict this limitation. This is a major drawback of the linear method. Moreover, certain methods of stabilization cannot be investigated using the linear method but only by a general non-linear analysis of the servocontrol. The method given below is an application of the results derived in Sections 6.6.3 and 6.10.

The basic idea is as follows. We assume that when there is no input, the servocontrol oscillates sinusoidally at the frequency corresponding to ω_c and with an amplitude z_0.

If the energy absorbed in a cycle is less than that received from the liquid, the amplitude z_0 will increase. The servocontrol will be unstable.

Note that we have assumed a *sinusoidal* oscillation. This limits the validity of the results to the case where the servocontrol takes up a motion which is close to sinusoidal when it is displaced from its equilibrium position, i.e. that where the non-linear forces (e.g. friction) are small compared with the linear forces (e.g. inertia force).

In Section 7.4.1. a certain number of restricting assumptions and approximations was made for the linear analysis; in particular we assumed that the *gain of the control valve was constant*, which is true only if the pressure in the two chambers remains in the region of $P_1/2$. This assumption cannot be made at high frequencies when the amplitude is not extremely low. If the assumption were retained, there would be no point in making the non-linear analysis. We will therefore reject it and compare the work *effectively* done on the servocontrol by the fluid with the energy effectively absorbed.

Finally, the use of a curve, or of a family of curves, plotted on reduced coordinates will enable all the calculations to be replaced by simple graphical constructions.

7.6.2. DETERMINATION OF THE WORK DONE BY THE FLUID

7.6.2.1. *For a control valve with a linear curve*

A valve with a linear curve is one which produces restriction areas for the flow which are directly proportional to the opening displacement, the constant of proportionality being the same for all four restrictions.

The flow equations—We will continue to define the valve by its gain: $K = Q_A/e = -Q_B/e$ when the loss of head in each of the restrictions is $P_1/2$ (cf. Section 7.4.2).

If we now wish to take into account the variations of the pressures P_A and P_B in the two chambers of the ram, we must take the flows as proportional to the square root of the pressure difference actually applied (i.e. (P_1-P_A), P_A, (P_1-P_B) or P_B) and not to the square root of $P_1/2$.

We therefore have the following expressions for Q_A and Q_B (see e.g. *Figure 7.3*):

if $e > 0$ (chamber A connected to P_1, chamber B to the return)

$$Q_A = \quad K e \sqrt{\frac{P_1 - P_A}{P_1/2}}$$

$$Q_B = - K e \sqrt{\frac{P_B}{P_1/2}}$$

if $e < 0$ (chamber B connected to P_1, chamber A to the return)

$$Q_A = \quad K e \sqrt{\frac{P_A}{P_1/2}}$$

$$Q_B = - K e \sqrt{\frac{P_1 - P_B}{P_1/2}}$$

The equations of continuity are

$$Q_A = S \frac{dz}{dt} + \frac{V_t}{B} \frac{dP_A}{dt} \quad , \quad Q_B = - S \frac{dz}{dt} + \frac{V_t}{B} \frac{dP_B}{dt}$$

Substituting for Q_A and Q_B, for $e > 0$ and for $e < 0$

$$S \frac{dz}{dt} = K e \sqrt{\frac{P_1 - P_A}{P_1/2}} - \frac{V_t}{B} \frac{dP_A}{dt} \equiv K e \sqrt{\frac{P_B}{P_1/2}} + \frac{V_t}{B} \frac{dP_B}{dt}$$

$$S \frac{dz}{dt} = K e \sqrt{\frac{P_A}{P_1/2}} - \frac{V_t}{B} \frac{dP_A}{dt} \equiv K e \sqrt{\frac{P_1 - P_B}{P_1/2}} + \frac{V_t}{B} \frac{dP_B}{dt}$$

With the initial conditions $P_A = P_B = P_1/2$, these relationships are satisfied if $P_B = P_1 - P_A$, a predictable result which indicates that the pressures P_A and P_B are 'symmetrical' with respect to $P_1/2$. This can be written

$$P_A = \frac{P_1}{2} + \frac{P_A - P_B}{2} \qquad\qquad P_B = \frac{P_1}{2} - \frac{P_A - P_B}{2}$$

from which the flow equations are

for $e > 0$	for $e < 0$
$Q_A = K e \sqrt{1 - \frac{P_A - P_B}{P_1}}$	$Q_A = K e \sqrt{1 + \frac{P_A - P_B}{P_1}}$
$Q_B = - K e \sqrt{1 - \frac{P_A - P_B}{P_1}}$	$Q_B = - K e \sqrt{1 + \frac{P_A - P_B}{P_1}}$

These relationships can be expressed by the following single equation, where $(x-z)/\lambda$ has been substituted for e:

$$Q_A = -Q_B = \frac{K}{\lambda}(x-z)\sqrt{1-\varepsilon\frac{(P_A-P_B)}{P_1}} \tag{17}$$

where

$$\begin{aligned} \epsilon &= +1 \quad \text{if} \quad (x-z) > 0 \\ &= -1 \quad \text{if} \quad (x-z) < 0 \end{aligned}$$

The 'discontinuity' at $e = 0$ corresponds to the change in 'layout' caused by the control valve: the chamber which was connected to P_1 becomes connected to the return and vice versa.

Equation for the work done—To simplify the presentation, we will calculate the work done during a half cycle which, because of the symmetry, is equal to half the work done in a complete cycle, W. With the assumptions of Section 7.6.1, $\omega = \omega_c$, $x = 0$, $z = z_0 \sin \omega_c t$, we have

$$\frac{W}{2} = \int_0^{\pi/\omega_c} (P_A Q_A + P_B Q_B)\,\mathrm{d}t \tag{18}$$

and, choosing the half cycle corresponding to $z > 0$ $(e < 0)$

$$Q_A = -Q_B = -\frac{K}{\lambda} z_0 \sin \omega_c t \sqrt{1+\frac{P_A-P_B}{P_1}} \tag{19}$$

Substituting this in eqn. (18)

$$\frac{W}{2} = \int_0^{\pi/\omega_c} Q_A(P_A-P_B)\,\mathrm{d}t = \int_0^{\pi/\omega_c} -\frac{K}{\lambda} z_0 \sin \omega_c t \sqrt{1+\frac{P_A-P_B}{P_1}}(P_A-P_B)\,\mathrm{d}t$$

N.B. For the linear approximation we have

$$\frac{W_l}{2} = \int_0^{\pi/\omega_c} -\frac{K}{\lambda} z_0 \sin \omega_c t\,(P_A-P_B)\,\mathrm{d}t$$

Determination of P_A-P_B—In Section 7.4 it was shown that

$$(P_A-P_B)S = rz + f\frac{\mathrm{d}z}{\mathrm{d}t} + m\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}$$

with $z = z_0 \sin \omega_c t$, this becomes

$$(P_A-P_B)S = mz_0\left[\left(\frac{r}{m}-\omega_c^2\right)\sin \omega_c t + \frac{f}{m}\omega_c \cos \omega_c t\right]$$

Now $\omega_c^2 = r_h/m \gg r/m$: hence r/m is negligible compared with ω_c^2. On the other hand, from Section 7.5.2 we have

$$f = \frac{2\,\zeta\, r_h}{\omega_c}$$

hence
$$\frac{f}{m}\,\omega_c = 2\,\zeta\,\omega_c^2$$

Substituting $\omega_c^2 = 2BS^2/k_v S z_M m$ gives

$$P_A - P_B \simeq -2\,B\,\frac{1}{k_v}\,\frac{z_0}{z_M}\,(\sin\omega_c t - 2\,\zeta\,\cos\omega_c t) \qquad (20)$$

By putting*

the critical amplitude $= z_c = z_M k_v P_1/2B$
the amplitude coefficient $= \alpha = z_0/z_c$
$\omega_c t = \theta$

and using the two definitions given in Sections 7.5.1 and 7.5.2

$$\begin{cases} r_h = \dfrac{2\,B\,S^2}{V_t} = \dfrac{2\,B\,S}{k_v z_M} \\ \omega_f = \dfrac{K}{\lambda\,S} \end{cases}$$

we have

$$W = 2\,z_0^2\,r_h\,\frac{\omega_f}{\omega_c}\int_0^{\pi}\sin\theta\,(\sin\theta - 2\,\zeta\cos\theta)\,\sqrt{1-\alpha\,(\sin\theta - 2\,\zeta\cos\theta)}\,d\theta \qquad (21)$$

We have assumed that the damping is very small: $\zeta \ll 1$. Now, since $\alpha = z_0/z_c \ll 1$, the term $2\alpha\zeta\cos\theta$ can be neglected compared with unity without introducing any great error. The equation for the work done then becomes

$$W \simeq 2\,z_0^2\,r_h\,\frac{\omega_f}{\omega_c}\int_0^{\pi}\sin\theta\,(\sin\theta - 2\,\zeta\,\cos\theta)\,\sqrt{1-\alpha\sin\theta}\,d\theta$$

In this equation, the integral

$$\int_0^{\pi} 2\,\zeta\,\sin\theta\,\cos\theta\,\sqrt{1-\alpha\sin\theta}\,d\theta$$

is zero since

$$\int_0^{\pi/2}\sin 2\,\theta\,\sqrt{1-\alpha\sin\theta}\,d\theta = -\int_{\pi/2}^{\pi}\sin 2\,\theta\,\sqrt{1-\alpha\sin\theta}\,d\theta$$

* Note the simple relationship between r_h and z_c: $r_h z_c = P_1 S = F_M$ which indicates that, in an oscillation with the valve closed, the pressures P_A and P_B are zero for $z = -z_c$ and $z = +z_c$.

The approximate equation for the work done by the fluid is therefore:

$$\boxed{W \simeq 2\, z_0^2 r_h \frac{\omega_f}{\omega_c} \int_0^\pi \sin^2\theta \sqrt{1 - \alpha \sin\theta}\, d\theta} \tag{22}$$

This equation contains an elliptic integral which can be evaluated as a function of the single parameter α.

But it is preferable to refer to the work done in the linear case, W_l, and to use this as a measure of W. To find W_l, put $\alpha = 0$ in eqn. (21)

$$W_l = 2\, z_0^2 r_h \frac{\omega_f}{\omega_c} \int_0^\pi \sin\theta\,(\sin\theta - 2\,\zeta \cos\theta)\, d\theta$$

Integration gives

$$W_l = \pi\, z_0^2 r_h \frac{\omega_f}{\omega_c}$$

We now define a coefficient β as the ratio of W to W_l

$$\boxed{\beta = \frac{W}{W_l} = \frac{2}{\pi} \int_0^\pi \sin^2\theta \sqrt{1 - \alpha \sin\theta}\, d\theta} \tag{23}$$

β depends only on α, i.e. *on z_0.*
There is a unique curve $\beta = f(\alpha)$ which covers all practical cases.
The following table gives some numerical results for β:

α	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
β	0·957	0·912	0·863	0·812	0·744	0·698	0·631	0·556	0·465	0·322

It will be shown later that in practice it is more convenient to plot $\alpha\beta = f(\alpha)$. This is represented by curve 1 in *Figure 7.8*. Note that:

$$W = \beta \cdot W_l = \alpha\beta \frac{W_l}{\alpha} = \alpha\beta \cdot \pi\, z_0 z_c r_h \frac{\omega_f}{\omega_c} \tag{24}$$

Note—The curve ends at $\alpha = 1$. To extend it beyond that, using the same equation, would imply that the pressure in the chambers can be reduced below the reservoir pressure, which is impossible if this is zero, or almost zero.

7.6.2.2. *A control valve with a dead zone**

This method can be extended to servocontrols with a valve having a non-linear curve. We will not deal with the general problem but will take the specific

* Not to be confused with the control valve having leakage flow, whose analysis can be made using linear methods and which will be dealt with below (Section 7.7.1.2).

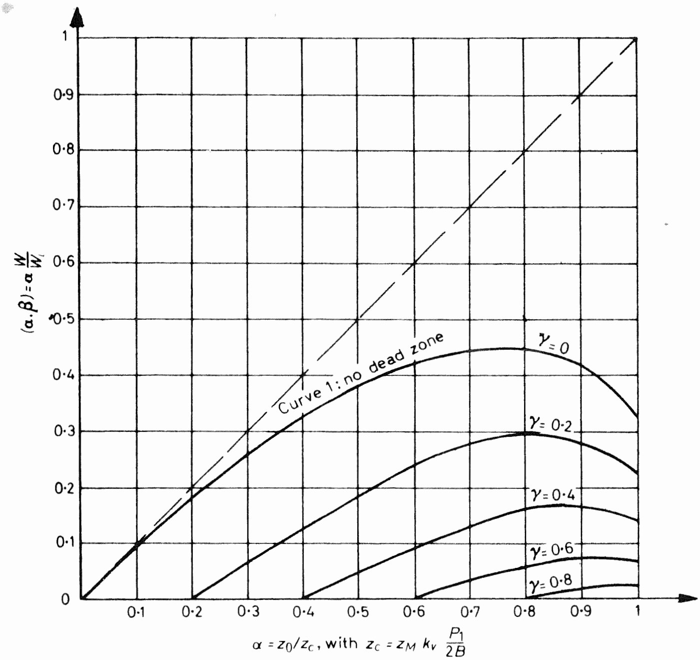


Figure 7.8

case of a control valve whose curve is linear with the exception of a dead zone. Suppose that the dead zone exists between $\pm s_0$.

Determination of the flows Q_A and Q_B—The equation $Q_A = -Q_B = -Kz/\lambda$ becomes

$$Q_A = Q_B = 0 \qquad \text{for} \quad z < \lambda s_0 \left(\sin \omega_c t < \frac{\lambda s_0}{z_0}\right)$$

$$\left.\begin{aligned} Q_A &= -\frac{K}{\lambda}(z - \lambda s_0)\sqrt{\frac{P_A}{P_1/2}} \\ Q_B &= \frac{K}{\lambda}(z - \lambda s_0)\sqrt{\frac{P_1 - P_B}{P_1/2}} \end{aligned}\right\} \quad \text{for} \quad z > \lambda s_0 \left(\sin \omega_c t > \frac{\lambda s_0}{z_0}\right)$$

A similar procedure to that used in Section 7.6.2.1 gives

$$Q_A = -Q_B = -\frac{K}{\lambda} z_0 \left(\sin \omega_c t - \frac{\lambda s_0}{z_0}\right)\sqrt{1 + \frac{(P_A - P_B)}{P_1}} \tag{25}$$

Defining a further reduced coefficient by $\gamma = \lambda s_0/z_0$ in order to define the extent of the dead zone, it can be shown that

$$W = 2 z_0^2 r_h \frac{\omega_f}{\omega_c} \int_{\sin^{-1}\gamma}^{\pi - \sin^{-1}\gamma} (\sin\theta - \gamma)(\sin\theta - 2\zeta\cos\theta)\sqrt{1 - a(\sin\theta - 2\zeta\cos\theta)}\, d\theta \tag{26}$$

As in the preceding Section, $2\alpha\zeta \cos\theta$ is negligible compared with unity.

In the new equation the two integrals

$$\int_{\sin^{-1}\gamma}^{\pi - \sin^{-1}\gamma} \sin 2\theta \sqrt{1 - a \sin\theta}\, d\theta$$

$$\int_{\sin^{-1}\gamma}^{\pi - \sin^{-1}\gamma} \cos\theta \sqrt{1 - a \sin\theta}\, d\theta$$

are zero.

The coefficient $\beta = W/W_l$ is a function of α and γ and is given by the approximate equation

$$\beta = \frac{2}{\pi} \int_{\sin^{-1}\gamma}^{\pi - \sin^{-1}\gamma} \sin\theta (\sin\theta - \gamma) \sqrt{1 - a \sin\theta}\, d\theta \qquad (27)$$

The curves $\alpha\beta = f(\alpha)$ for different values of γ are given in *Figure 7.8*. Curve 1, which was dealt with above, simply corresponds to the particular case $\gamma = 0$ (no dead zone).

The work done, W, is always given by the equation

$$W = a\beta \frac{W_l}{a} = a\beta \cdot \pi z_0 z_c r_h \frac{\omega_f}{\omega_c} \qquad (24)$$

7.6.3. DETERMINATION OF ENERGY ABSORBED

7.6.3.1. *Viscous friction*

Coefficient of viscous friction: f

Frictional force: $F = f\, dz/dt = f z_0 \omega_c \cos \omega_c t$

Work done: $dW_a = F\, dz = f z_0^2 \omega_c^2 \cos^2 \omega_c t\, dt$

Energy absorbed in one cycle

$$W_a = \int_0^{2\pi/\omega_c} f \omega_c^2 z_0^2 \cos^2 \omega_c t\, dt = \pi f \omega_c z_0^2 \qquad (28)$$

(For a frequency $\omega_i \neq \omega_c$, $W_a = \pi f \omega_i z_0^2$.)

7.6.3.2. *Energy associated with the reduced damping coefficient, ζ, of the transfer function*

In Section 7.5.2 it was shown that

$$\zeta = \frac{\omega_c}{2} \frac{f}{r_h} \quad \text{so that} \quad f = \frac{2 r_h \zeta}{\omega_c}$$

Substituting this in eqn. (28) gives

$$W_a = 2\pi r_h \zeta z_0^2 \qquad (29)$$

(For a frequency $\omega_i \neq \omega_c$, $W_a = 2\pi r_h \zeta z_0^2 \omega_i/\omega_c$.)

7.6.3.3. *Coulomb friction*

Constant frictional force, F_f. Work done against friction per cycle

$$W_a = 4\, z_0\, F_f \tag{30}$$

(W_a is independent of the frequency.)

7.6.4. PRACTICAL METHOD OF DETERMINING THE STABILITY

In Section 7.6.2 it was shown that the work done by the fluid, W, is the product of two terms:

(1) the dimensionless coefficient $\alpha\beta$, which depends on α and γ and can be read directly off *Figure 7.8*;
(2) the dimensional term $\pi z_0 z_c r_h \omega_f/\omega_c$.

Instead of determining W and comparing it with the energy absorbed, W_a, it is much simpler to find

$$w_a = \frac{W_a}{\pi\, z_0\, z_c\, r_h\, \omega_f/\omega_c}$$

and to make the comparison directly on the graphs.

Table 7.2 gives the expressions for the reduced energy, w_a, for direct comparison with the curves $\alpha\beta = f(\alpha)$.

Table 7.2

	W_a	w_a	*Representation of w_a as a function of α*
Viscous friction, $F = f\,\mathrm{d}z/\mathrm{d}t$	$\pi f \omega_c z_0^2$	$\dfrac{f}{m\omega_f}\dfrac{z_0}{z_c}$ (in fact $\omega_c^2 = r_h/m$)	straight line through the origin
Friction associated with the reduced coefficient ζ	$2\pi r_h \zeta z_0^2$	$2\zeta\dfrac{\omega_c}{\omega_f}\dfrac{z_0}{z_c}$	(straight line through the origin)
Coulomb friction, F_f	$4F_f z_0^2$	$\dfrac{4}{\pi}\dfrac{F_f}{F_M}\dfrac{\omega_c}{\omega_f}$ (in fact: $z_c r_h = P_1 S = F_M$)	horizontal line
Any combination of the three preceding cases			straight line through the point $\alpha = 0$ $\alpha\beta = \dfrac{4}{\pi}\dfrac{F_f}{F_M}\dfrac{\omega_c}{\omega_f}$

We can now appreciate the reason for the method of presentation of the curves in Figure 7.8.

The determination of stability is made by plotting straight lines on a family of curves which are fixed for all problems and all unknowns.

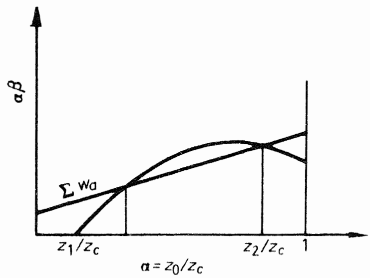


Figure 7.9

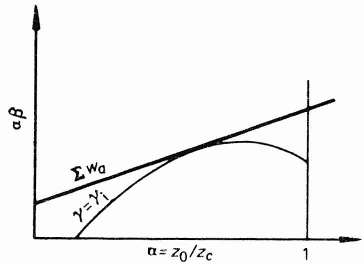


Figure 7.10

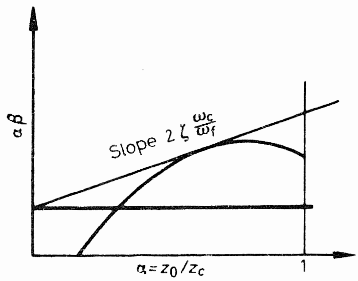


Figure 7.11

Example

(1) *To determine whether an existing system is stable* (*Figure 7.9*)—Plot the straight line Σw_a representing the total absorbed energy and compare it with the $\alpha\beta$ curve corresponding to the value of γ for the existing control valve. If the two lines do not intersect, the servocontrol is stable. If there is an intersection, the graph indicates the amplitude limit beyond which the oscillations are amplified (z_1 on *Figure 7.9*) and the amplitude which will be attained (z_2).

(2) *To determine the valve dead zone necessary to stabilize a servocontrol*—To determine γ, find the $\alpha\beta$ curve which just touches the line Σw_a (*Figure 7.10*).

(3) *To determine the artificial damping ζ necessary to stabilize a servocontrol of given dead zone and given friction* (*Figure 7.11*)—Draw the tangent to the $\alpha\beta$ curve corresponding to the value of γ of the valve through the point:

$$\alpha = 0, \qquad \alpha\beta = \frac{4}{\pi}\frac{F_f}{F_M}\frac{\omega_c}{\omega_f}$$

The slope of this tangent is equal to $2\zeta\omega_c/\omega_f$.

Notes

(1) For a servocontrol without a dead zone or friction, the condition for stability is (see *Figure 7.8*)

$$\frac{f}{m\omega_f}\frac{z_0}{z_c} > \frac{z_0}{z_c} \quad \text{i.e.} \quad \frac{f}{m\omega_f} > 1 \quad \text{or} \quad f > m\omega_f$$

or in terms of ζ

$$2\zeta\frac{\omega_c}{\omega_f}\frac{z_0}{z_c} > \frac{z_0}{z_c} \quad \text{i.e.} \quad 2\zeta\frac{\omega_c}{\omega_f} > 1 \quad \text{or} \quad \zeta > \frac{1}{2}\frac{\omega_f}{\omega_c}$$

These conditions have already been obtained in Section 7.5.3.

In this case, rejection of the simplifying assumption of linearity does not change the absolute condition for stability. The only advantage is that the amplitude of the unstable oscillations can be determined. On the other hand, the new method is very useful if the control valve has a dead zone or if the Coulomb friction force is not negligible.

(2) Note that the graphical constructions given above determine the absolute conditions for stability. It is advisable to allow a safety margin in addition.

(3) Note that the method is only valid for a fixed servocontrol. Also, that a servocontrol with a dead zone can be stable for a fixed input and unstable for an input corresponding to a high velocity such that the valve operates far from its dead zone. The analysis for continuous motion can then be made using linear methods.

Generalization for the case where the only non-linearity is due to Coulomb friction, F_f—In addition to the frequency $\omega = \omega_c$, interesting information can be gained by comparing the actual system to the linear tangent system (same energy absorbed) for each value ω_i of ω and for each value z_i of z_0. It is particularly useful to continue using eqn. (7) and (8) while considering f and ζ as functions of w_i and z_i:

$$f(\omega_i, z_i) = \frac{4}{\pi} F_f \frac{1}{\omega_i z_i}$$

$$\zeta(\omega_i, z_i) = \frac{\omega_c}{2} \frac{V_t}{2BS^2} \frac{4}{\pi} F_f \frac{1}{\omega_i z_i}$$

This method is obviously valid only for motion which is approximately sinusoidal, i.e. sinusoidal input and low values of $\zeta(w_i, z_i)$.

7.7. SOME METHODS OF STABILIZATION

In order to stabilize a servocontrol without affecting its performance, we must artificially increase the damping coefficient ζ of the transfer function to the value given in Section 7.5.3.

If the stabilization process is linear, we can analyse it by making the simplifying linear assumption for the servocontrol itself, since it was shown, at the end of the last Section, that this does not affect the final result. If the stabilization process cannot be represented by linear equations, we have to use the method given in Section 7.6.

To classify the stabilization processes, we can divide them into those which require throttling of the liquid, often called *leakage stabilization* methods; those which require either *secondary detection* or a *particular valve curve*, and *complementary* processes which are theoretically insufficient but which complete the preceding methods: the use of a dash-pot (low-pass filter).

7.7.1. STABILIZATION BY LEAKAGE FLOW

The leakage flow between two chambers at different pressures is accompanied by a loss of energy without any external work being done. As in the case of friction, the energy is dissipated as heat. It is to be expected, then, that the careful addition of leakage flow to a system can have a stabilizing effect.

In practice, two types of leakage flow are used, either between the two chambers of the ram or past the valve spool. We will examine these two methods.

7.7.1.1. *Leakage between the two chambers of the ram*

Laminar leakage flow (*Figure 7.12*)—Suppose that there is an orifice connecting the two chambers of the ram, either in the piston or between the supply pipes,

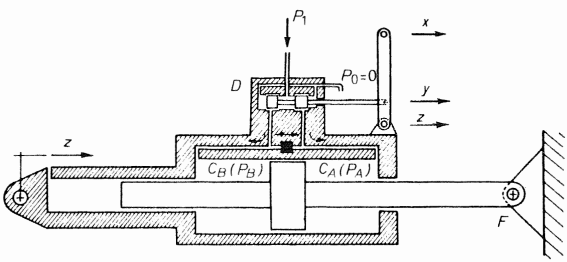


Figure 7.12

such that the flow through the orifice is proportional to the loss of head* (laminar orifice or orifice with linear characteristic; cf. Section 3.2.2.2).

Let A be the flow coefficient for the orifice (flow per unit pressure difference):

$$A = \frac{q\ (\text{cm}^3/\text{sec})}{\Delta P\ (\text{kg/cm}^2)}$$

The equations

$$Q_A = K e = \frac{K}{\lambda}(x - z) = S\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{V_t}{B}\frac{\mathrm{d}P_A}{\mathrm{d}t} \tag{2}$$

$$Q_B = -K e = \frac{-K}{\lambda}(x - z) = -S\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{V_t}{B}\frac{\mathrm{d}P_B}{\mathrm{d}t} \tag{3}$$

become

* The manufactured clearance between piston and cylinder cannot be relied on to ensure a stabilizing leakage flow. It varies from one article to the next.

$$Q_A = \quad K e = \quad \frac{K}{\lambda}(x - z) = \quad S \frac{dz}{dt} + \frac{V_t}{B}\frac{dP_A}{dt} + A(P_A - P_B)$$

$$Q_B = -K e = -\frac{K}{\lambda}(x - z) = -S \frac{dz}{dt} + \frac{V_t}{B}\frac{dP_B}{dt} - A(P_A - P_B)$$

wherefrom

$$\frac{K}{\lambda}(x - z) = S p z + (P_A - P_B)\left(A + \frac{V_t}{2B} p\right) \tag{33}$$

Eliminating $(P_A - P_B)$ between eqn. (33) and $(P_A - P_B)S = z(r + fp + mp^2)$ gives

$$H_1 = \frac{z}{x - z} = \frac{K/\lambda S}{\frac{Ar}{S^2} + p\left[1 + \frac{V_t r}{2BS^2} + \frac{fA}{S^2} + \left(\frac{Am}{S^2} + \frac{V_t f}{2BS^2}\right)p + \frac{V_t m}{2BS^2}p^2\right]} \tag{34}$$

It was shown (Section 7.5) that $V_t r/2BS^2$ is negligible compared with unity.

Putting (cf. Section 7.5.2)

$$\omega_f = \frac{K}{\lambda S} \quad ; \quad \omega_c = \sqrt{\frac{2BS^2}{V_t m}} \quad ; \quad \zeta = \frac{\omega_c}{2}\frac{V_t f}{2BS^2} \qquad \text{(natural damping)}$$

defining the new quantities

$$\varepsilon_1 = \frac{Ar}{S^2}$$

$$\zeta_1 = \frac{\omega_c}{2}\frac{Am}{S^2} \qquad \text{(artificial damping)}$$

and noting that

$$\frac{fA}{S^2} = \frac{2\zeta_1}{\omega_c m} \cdot \frac{4\zeta B S^2}{V_t \omega_c} = 4\zeta\zeta_1$$

eqn. (33) becomes

$$H_1 = \frac{\omega_f}{\varepsilon_1 + p\left[1 + 4\zeta\zeta_1 + 2(\zeta + \zeta_1)\frac{p}{\omega_c} + \frac{p^2}{\omega_c^2}\right]} \tag{35}$$

ζ is generally very small and ζ_1, which is limited by considerations of fluid consumption, rarely exceeds 0·3; $4\zeta\zeta_1$ can therefore generally be neglected, so that

$$\boxed{H_1 = \frac{\omega_f}{\varepsilon_1 + p\left[1 + 2(\zeta + \zeta_1)\frac{p}{\omega_c} + \frac{p^2}{\omega_c^2}\right]}} \tag{35 }$$

The leakage A introduces a supplementary *damping term*, ζ_1, into the open loop but, at the same time, a proportional constant term, ϵ_1, which causes a *static position error* in the closed loop. It was shown (Section 6.8) that, if the closed-loop gain is fairly high, this error is inversely proportional to the gain, so that

$$\frac{\Delta z_0}{z_0} = \frac{\varepsilon_1}{\omega_f} = \frac{A\,r}{S^2\,\omega_f} = \frac{A\,r\,\lambda}{K\,S}$$

and denoting the opposing force*, rz_0, by F_0

$$\Delta z_0 = \frac{A\,F_0\,\lambda}{K\,S} \tag{36}$$

Note 1—If S has not been determined, it can be more convenient to put ζ_1, ϵ_1 and Δz_0 in the following forms, obtained by using the relation $\omega_c^2 = 2BS^2/V_t m$:

$$\zeta_1 = \frac{A}{\omega_c}\,\frac{B}{V_t} \tag{37}$$

$$\varepsilon_1 = \frac{A}{\omega_c^2}\,\frac{2\,B}{V_t}\,\frac{r}{m} = \frac{2}{\omega_c}\,\frac{r}{m}\,\zeta_1 \tag{37'}$$

$$\Delta z_0 = \frac{A}{\omega_c^2}\,\frac{2\,B}{V_t}\,\frac{F_0}{m\,\omega_f} \tag{37''}$$

Note 2—In dynamic analysis of the servocontrol (gain regulation, determination of the closed-loop resonant frequency, stability analysis), the approximate equation can be used

$$H_1 = \frac{\omega_f}{p\left[1 + 2(\zeta + \zeta_1)\,\dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right]}$$

since ϵ_1 is small and can be neglected in comparison with the other terms of the transfer function.

For small values of ω, however, H_1 is better represented by

$$H_1 = \frac{\omega_f}{\varepsilon_1 + p}$$

so that again there is no integration in the open loop.

---

* The existence of a position error proportional to the opposing force can be accounted for without mathematical analysis. The opposing force is balanced by the pressure difference between the two chambers of the ram which causes a flow through the orifice. This means that the valve must be open to a certain extent to allow for this flow, i.e. *there is a non-zero error*.

As ω decreases, Nichols curves (*Figure 7.6*) no longer tend to infinity but to the point (ω_f/ϵ_1 dB; 0°)*.

Determination of the minimum leakage flow necessary, A_n, and the corresponding static error—If the leakage is the only stabilizing factor, and if $\omega_f/\omega_c \leqslant 0{\cdot}35$, the minimum damping coefficient necessary has been shown (Section 7.5.3) to be $\zeta_n = \omega_f/\omega_c$, and the minimum leakage necessary is

$$A_n = \omega_f \frac{V_t}{B} \tag{38}$$

The corresponding term, ϵ_1, is given by

$$\varepsilon_{1n} = \omega_f \frac{V_t r}{B S^2} = \omega_f k_v \frac{P_1}{B} \frac{r z_M}{F_M}$$

and the static position error by

$$\Delta z_{0_n} = \frac{\varepsilon_{1n} z_0}{\omega_f} = k_v \frac{P_1}{B} \frac{F_0}{F_M} z_M$$

If the opposing stiffness, r, is constant,

$$r z_M = \frac{F_M}{k_s} \qquad \frac{F_0}{F_M} = \frac{z_0}{z_M}$$

so that we have

$$\varepsilon_{1n} = \omega_f \frac{k_v}{k_s} \frac{P_1}{B} \qquad \Delta z_{0_n} = \frac{k_v}{k_s} \frac{P_1}{B} z_0$$

If the servocontrol already contains a linear stabilizing factor ($\zeta \neq 0$)

$$\zeta_n = \frac{\omega_f}{\omega_c} - \zeta$$

so that

$$A_n = (\omega_f - \zeta \omega_c) \frac{V_t}{B}$$

If the servocontrol contains a non-linear stabilizing factor (Coulomb friction or a dead zone), we can use the method described in Section 7.6 and summarized in Section 7.6.4.

Turbulent leakage flow—Turbulent flow through the orifice is much more common than laminar flow. The analysis is difficult since, as we have seen, Bernoulli's equation is not linearizable near the origin.

To facilitate the analysis we will use the results derived in the previous Section. We will assume that a laminar orifice of flow coefficient A corresponds

* An example of such a curve is given in *Figure 7.39* (Appendix 7.1) by curve F_2 which corresponds to $\omega_f = \frac{1}{4}\omega_c$; $\zeta = 0{\cdot}1$; $\zeta_1 = 0{\cdot}25$; $\epsilon_1 = 0{\cdot}5 \times 10^{-2}\omega_c$.

to a turbulent orifice which has the same flow under $\Delta P = P_1$, so that $q_1 = AP_1$. The flow through this orifice

$$q = K\sqrt{|\Delta P|} \cdot \text{ sign of } \Delta P$$

is therefore

$$q = A\sqrt{P_1}\sqrt{|\Delta P|} \cdot \text{ sign of } \Delta P$$

The two orifices which are associated in this way are known as 'corresponding orifices'.

The theory of equivalent linear equations (Section 5.5.2) shows that for a sinusoidal variation of ΔP of amplitude ΔP_0, we have for the turbulent orifice

$$q = 1{\cdot}113\sqrt{\frac{P_1}{\Delta P_0}}\,A\,\Delta P \tag{39}$$

instead of $q = A/\Delta P$ for the linear orifice. The reduced damping coefficient, ζ_1, and the energy absorbed, W, are therefore changed by the factor $1{\cdot}113\sqrt{P_1/\Delta P_0}$ when a turbulent orifice is used instead of the corresponding linear one.

This enables us to determine easily the size of the turbulent orifice necessary. Consider the graph $\alpha\beta = f(\alpha, \gamma)$ of *Figure 7.8* which has been reproduced in *Figure 7.13.*

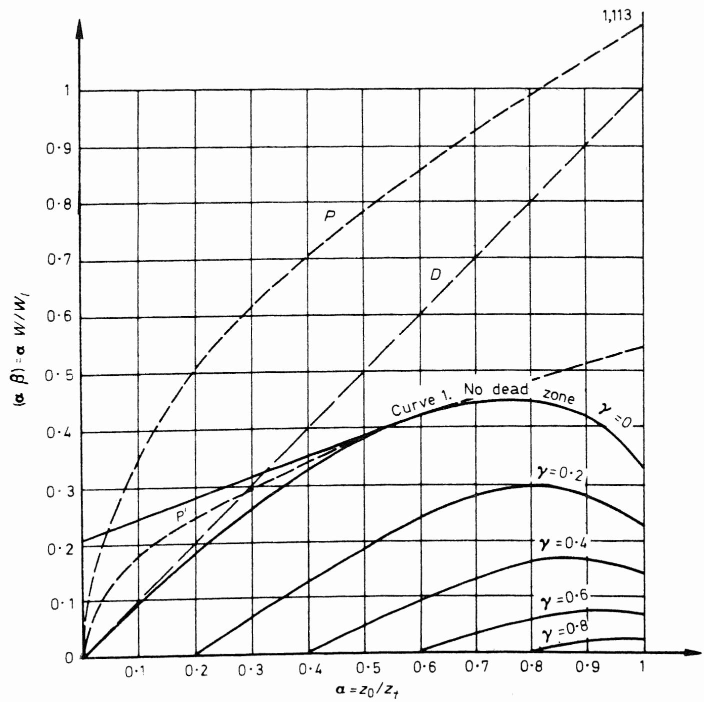


Figure 7.13

Suppose that the orifice necessary for stabilization is the linear orifice of flow coefficient A which is represented by the straight line D. The corresponding turbulent orifice is represented by the parabola P which passes through $\alpha = 1$, $\alpha\beta = 1{\cdot}113$. For $\omega = \omega_c$ and $z_0 = z_c$, $\Delta P_0 = P_1$—the orifice is much larger than is necessary.

A reduction by a factor of 0·485 results in parabola P' which is tangential to curve 1. We conclude that the turbulent orifice necessary for the stabilization of a servocontrol which has no other stabilizing factors is that which, for a loss of head of P_1, has a flow equal to 0·485 times that of the necessary laminar orifice determined (note that this ratio is approximately 0·5).

If the servocontrol has other stabilizing factors, the orifice necessary can be determined by an analogous method. Plot the straight line representing the minimum laminar orifice necessary, the parabola representing the corresponding turbulent orifice and finally the similar parabola tangential to the curve $\alpha\beta = f(\alpha)$ of the servocontrol.

7.7.1.2. *Control valve leakage flow**

Up to now we have considered servocontrols whose control valve provided a complete seal when in the closed position. However, there are valves which are designed to give flow even in the closed position. The conditions under which the piston of the ram remains stationary are simply (see *Figure 7.14*)

$$\begin{cases} q_1 = q_2 \\ q_3 = q_4 \end{cases}$$

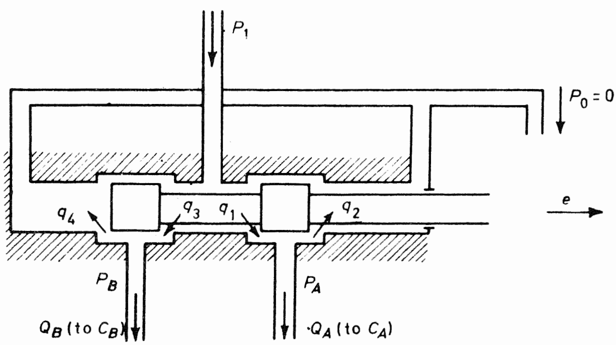


Figure 7.14

In the following analysis we will assume that the valve is completely symmetrical and that the four restriction areas, s_1, s_2, s_3 and s_4, vary linearly with e. They are therefore all equal in the neutral position ($e = 0$).

Let this area be s_0 and let q_0 be the flow through s_0 under a pressure difference $\Delta P = P_1/2$. Note that the permanent flow in the neutral position is therefore $2q_0$.

* The method given also applies to the stability analysis for a valve without leakage during a movement at constant control speed.

Deriving the equation—Let e_0 *be the value of e for which* s_2 *and* s_3 *are zero* ($-e_0$ is therefore the value at which s_1 and s_4 are zero). We have

$$s_1 = s_4 = s_0 \left(1 + \frac{e}{e_0}\right)$$

$$s_2 = s_3 = s_0 \left(1 - \frac{e}{e_0}\right)$$

Linearization of the flow gives

$$q_1 = C \cdot s_1 \sqrt{P_1 - P_A} = q_0 \left[1 + \frac{e}{e_0} + \frac{1}{2} \frac{\Delta (P_1 - P_A)}{(P_1 - P_A)_0}\right]$$

$$q_2 = C \cdot s_2 \sqrt{P_A} = q_0 \left[1 - \frac{e}{e_0} + \frac{1}{2} \frac{\Delta P_A}{P_{A0}}\right]$$

$$q_3 = C \cdot s_3 \sqrt{P_1 - P_B} = q_0 \left[1 - \frac{e}{e_0} + \frac{1}{2} \frac{\Delta (P_1 - P_B)}{(P_1 - P_B)_0}\right]$$

$$q_4 = C \cdot s_4 \sqrt{P_B} = q_0 \left[1 + \frac{e}{e_0} + \frac{1}{2} \frac{\Delta P_B}{P_{B0}}\right]$$

where C is a constant.

For small movements, P_A and P_B remain in the neighbourhood of $P_1/2$ and $(P_1-P_A)_0$, P_{A_0}, $(P_1-P_B)_0$ and P_{B_0} are equal to $P_1/2$. Finally,

$$\Delta(P_1 - P_A) = -\Delta P_A$$

so that we have

$$Q_A = q_1 - q_2 = q_0 \left[2 \frac{e}{e_0} - 2 \frac{\Delta P_A}{P_1}\right] \tag{40}$$

$$Q_B = q_3 - q_4 = q_0 \left[-2 \frac{e}{e_0} - 2 \frac{\Delta P_B}{P_1}\right] \tag{41}$$

Equating these to the previous expressions for Q_A and Q_B

$$Q_A = S \frac{dz}{dt} + \frac{V_t}{B} \frac{dP_A}{dt} \qquad Q_B = -S \frac{dz}{dt} + \frac{V_t}{B} \frac{dP_B}{dt}$$

Subtracting term by term and applying the Laplace transformation gives

$$\frac{2 q_0}{e_0} e = z S p + (P_A - P_B) \left(\frac{q_0}{P_1} + \frac{V_t}{2 B} p\right) \tag{42}$$

The gain of the valve, defined here as the increase in flow per unit displacement, is $2q_0/e_0$, so we have

$$K e = \frac{K}{\lambda} (x - z) = z S p + (P_A - P_B) \left(\frac{q_0}{P_1} + \frac{V_t}{2 B} p\right) \tag{43}$$

Elimination of $(P_A - P_B)$ between this equation and the force equation (Section 7.4) gives the transfer function

$$H_1 = \frac{K/\lambda S}{\dfrac{q_0 r}{P_1 S^2} + p\left[1 + \dfrac{V_t r}{2BS^2} + \dfrac{q_0}{P_1}\dfrac{f}{S^2} + \left(\dfrac{q_0 m}{P_1 S^2} + \dfrac{V_t f}{2BS^2}\right)p + \dfrac{V_t m}{2BS^2}p^2\right]} \tag{44}$$

Putting

$$\varepsilon_1' = \frac{q_0 r}{P_1 S^2} \qquad \zeta_1' = \frac{\omega_c}{2}\frac{q_0}{P_1}\frac{m}{S^2}$$

and, as in Section 7.5.2,

$$\omega_f = \frac{K}{\lambda S}, \quad \omega_c = \sqrt{\frac{2BS^2}{V_t m}} \qquad \zeta = \frac{\omega_c}{2}\frac{V_t f}{2BS^2}$$

with $V_t r/2BS^2$ negligible compared with unity as before and $q_0 f/P_1 S^2$ also usually negligible compared with unity, eqn. (44) becomes

$$\boxed{H_1 = \frac{\omega_f}{\varepsilon_1' + p\left[1 + 2(\zeta + \zeta_1')\dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right]}} \tag{45}$$

7.7.1.3. *Comparison of the two leakage processes*

The transfer function of eqn. (45) is exactly the same as that of eqn. (35′), which was derived for the servocontrol having a laminar orifice between the two chambers of the ram, with ϵ_1' replacing ϵ_1 and ζ_1' replacing ζ_1.

Furthermore, if we compare the terms

$$\varepsilon_1 = \frac{Ar}{S^2} \qquad \varepsilon_1' = \frac{q_0}{P_1}\frac{r}{S^2}$$

$$\zeta_1 = A\frac{\omega_c}{2}\frac{m}{S^2} \qquad \zeta_1' = \frac{q_0}{P_1}\frac{\omega_c}{2}\frac{m}{S^2}$$

we see that the *one transfer function can be derived from the other simply by substituting A for q_0/P_1.*

Note also that

$$\frac{\varepsilon_1}{\zeta_1} = \frac{\varepsilon_1'}{\zeta_1'} = \frac{2}{\omega_c}\frac{r}{m}$$

Comparison of static errors—As a result of the previous comments, it can be seen that leakage in the ram and leakage in the corresponding valve ($\zeta_1' = \zeta_1$) introduce the same static error ($\epsilon_1' = \epsilon_1$).

Comparison of permanent flows. We will compare the permanent flows of the following three corresponding cases

laminar leakage flow in the ram, q_l

turbulent leakage flow in the ram, q_t

leakage flow in the valve, q_d

and express them as a function of q_0, the leakage flow in one-half of the valve when in the closed position. For the cases to correspond, we must have $A = q_0/P_1$.

Determination of q_l:

$$q_l = A\,\Delta P = q_0 \frac{\Delta P}{P_1}$$

Determination of q_t:

$$q_t = k\sqrt{\Delta P}$$

When $\Delta P = P_1$, $q_t = 0{\cdot}485\, q_l$, $\therefore k\sqrt{P_1} = 0{\cdot}485\, q_0$ and

$$q_t = 0{\cdot}485\, q_0 \sqrt{\frac{\Delta P}{P_1}}$$

Determination of q_d: The linear approximation made above gives

$$q_d = \text{constant} = 2q_0$$

This approximation is a pessimistic one (since, in particular, q_d is zero for $e = \pm e_0$). A more accurate analysis gives

$$q_d = 2\,q_0 \frac{\sqrt{1-(\Delta P/P_1)^2}\,(\sqrt{1+\Delta P/P_1} - \sqrt{1-\Delta P/P_1})}{\Delta P/P_1}$$

$q_l/2q_0$, $q_t/2q_0$ and $q_d/2q_0$ are shown as functions of $\Delta P/P_1$ in *Figure 7.15*.

The advantage of the methods using leakage in the ram is obvious; moreover, the ram operates much more frequently under low loads (small values of ΔP) than under loads near the maximum ($\Delta P = P_1$), so that the advantage is even greater.

This theoretical comparison is important in practice, since stabilization leakage flow is not negligible in comparison with the mean flow.

Compare, for example, the stabilization leakage flow in a control valve with the flow of the servocontrol when oscillating at a frequency ω_0 with an amplitude $\lambda z_M (\lambda \ll 1)$. If there is no other source of damping

$$A_{\text{necessary}} = \frac{q_0}{P_1} = \omega_f \frac{V_t}{B}$$

—eqn. (38)—so that

$$q_d = 2\,q_0 = 2\,\omega_f V_t \frac{P_1}{B}$$

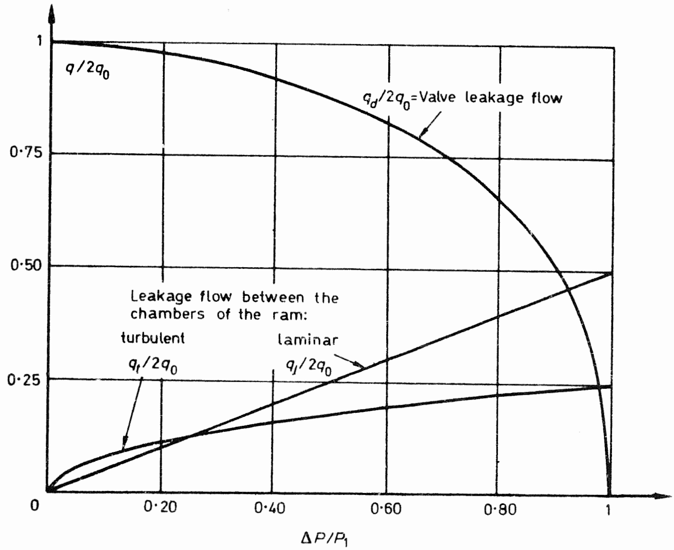


Figure 7.15

The permanent flow corresponds to four times the volume $\lambda z_M S$ per cycle and is therefore equal to

$$c = 4\,\lambda z_M\,S\,\frac{\omega_0}{2\pi} = \frac{2}{\pi}\,\lambda\,\frac{V_t}{k_v}\,\omega_0$$

This gives

$$\frac{q_d}{c} = \frac{\pi}{\lambda}\,k_v\,\frac{P_1}{B}\,\frac{\omega_f}{\omega_0}$$

For a ram supplied with oil of bulk modulus 15,000 kg/cm^2 at a pressure of 150 kg/cm^2, having a coefficient $k_v = 1{\cdot}2$ and oscillating at an angular frequency, ω_f, with an amplitude of $\frac{1}{10}$ the maximum amplitude, we therefore have

$$\frac{q_d}{c} = 0{\cdot}38$$

Under the same conditions, without making the complete calculation taking into account the variation of q_l and q_t in a cycle, it can be seen that q_l will be less than $q_0\Delta P_{\max}/P_1$ and q_t will be less than $0{\cdot}485 q_0\sqrt{\Delta P_{\max}/P_1}$. Thus, with

$$\Delta P_{\max} = \lambda P_1/k_s$$

$$\begin{cases} \dfrac{q_l}{2\,q_0} < 0{\cdot}5\,\dfrac{\lambda}{k_s} \\[2ex] \dfrac{q_t}{2\,q_0} < 0{\cdot}243\,\sqrt{\dfrac{\lambda}{k_s}} \end{cases}$$

If, for example, $k_s = 1{\cdot}2$ and $\lambda = 0{\cdot}1$

$$\left\{\begin{array}{l} \dfrac{q_l}{q_d} < 0{\cdot}041 \rightarrow \dfrac{q_l}{c} < 0{\cdot}015 \\[2ex] \dfrac{q_t}{q_d} < 0{\cdot}070 \rightarrow \dfrac{q_t}{c} < 0{\cdot}027 \end{array}\right.$$

7.7.2. *Stabilization by secondary feedback*

It has just been shown that the leakage stabilization method has a major disadvantage—the permanent flow required. We may well ask whether there are other processes which do not have this disadvantage; for example, the use of secondary feedback.

7.7.2.1. *Quantities suitable for detection*

Quantities which may be used for detection at the ram are:

the position, z, and its derivatives, zp^u, with $u = 0, 1, 2$, etc.

the pressure difference $(P_A - P_B)$ between the two chambers and its derivatives $(P_A - P_B)p^v$, with $v = 0, 1, 2$, etc.

The introduction of a secondary feedback chain for zp^u or $(P_A - P_B)p^v$ means that the feedback z (considered alone previously) is replaced by

$$z\,(1 + \beta\, p^u) \qquad \text{or} \qquad z + \gamma\,(P_A - P_B)\,p^v$$

where β and γ are two coefficients denoting the significance of the secondary feedback.

To simplify the procedure, we will introduce the two feedback chains simultaneously, i.e. the total feedback will be

$$z + \beta z p^u + \gamma (P_A - P_B) p^v$$

If we assume that the friction is negligible, eqn. (4) and (5) [p. 239] become

$$(P_A - P_B) = \frac{z}{S}\,(r + m\,p^2) \tag{52}$$

$$\frac{K}{\lambda}\left[x - z - \beta\, z\, p^u - \gamma\,(P_A - P_B)\,p^v\right] = p\, z\, S + \frac{V_t}{2\,B}\,p\,(P_A - P_B) \tag{53}$$

Elimination of $(P_A - P_B)$ between eqn. (52) and (53) gives the open-loop transfer function

$$H_1 = \frac{K/\lambda S}{\left(1 + \dfrac{V_t r}{2\,BS^2}\right) p + \dfrac{V_t m}{2\,BS^2}\,p^3 + \dfrac{K\beta}{\lambda S}\,p^u + \dfrac{K\gamma}{\lambda S^2}\,p^v\,(r + m\,p^2)} \tag{54}$$

To obtain a stabilizing term (in p^2) in the denominator we can make

$$u = 2$$

$$v = 0$$

or $$v = 2$$

$u = 2$ *indicates detection of the acceleration of the ram*

This detection is not easy to achieve in practice (although it has been done). This process, known as acceleration feedback, has the major theoretical advantage of not introducing a position error.

$v = 0$ *indicates detection of the pressure difference*

This is easy to arrange and is often used in practice. The process, known as pressure feedback, has the disadvantage of introducing the constant term $K\gamma r/\lambda S^2$ which results in a position error.

$v = 2$ *indicates detection of the second derivative of* $(P_A - P_B)$

To the best of the author's knowledge this type of detection cannot be achieved in practice. *Note that the derivative feedback is not effective here**.

7.7.2.2. *Introduction of detected quantities to the error signal*

Since the error, $x-z$, is a length, we will first consider introducing the feedback in the form of a length. This is easy in theory. For a servocontrol with direct connection ($\lambda = 1$), we could achieve secondary detection by using the relative movement between the valve body and the movable part of the ram. This solution would complicate the construction of the servocontrol, since it increases the number of seals necessary between movable parts.

It is better to use the relative movement of the movable part of the ram and the hinged joint, γ, of the connection lever (*Figure 7.16*).

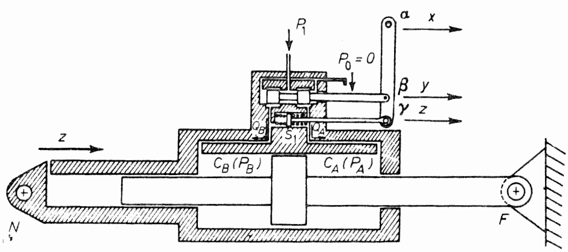


Figure 7.16

There is, however, another method of achieving the secondary detection: the valve aperture could be controlled directly by the unchanged input. If the actual control lever (i.e. the pilot's manual control lever, for example) is connected to point C of the control rod by a linkage system which is not perfectly rigid, a force applied to the valve spool could control the opening without changing the position of the control lever.

If the servocontrol is controlled by a flexible linkage system, it is possible to introduce the secondary detection in the form of a force applied at the valve stage (*Figure 7.17*). The magnitude of the force required to give a certain amount of secondary

* It can easily be shown that the derivative feedback only stabilizes systems whose open-loop transfer function is tangential to a first-order function or to a highly damped second-order function.

feedback obviously depends, other things being equal, on the flexibility of the control linkage.

7.7.2.3. *Pressure feedback* (see also Appendix 7.1)

We will examine pressure feedback using the two systems mentioned, pressure transformed either to position or to force.

Detected pressure transformed to position (see *Figure 7.16*)—Let

s_1 = area of the detection piston
E = total stiffness of the springs of the detection piston
t = movement of this piston relative to the ram.

The kinematic equation is

$$x - (z + t) = \lambda [y - (z + t)]$$

i.e.

$$y - z = \frac{x - z}{\lambda} + t \frac{\lambda - 1}{\lambda} \tag{55}$$

In addition, we have

$$t = -(P_A - P_B) \frac{s_1}{E} \tag{56}$$

Eliminating y, t and $(P_A - P_B)$ between eqn. (55), (56), (52) and (53) gives

$$H_1 = \frac{K/\lambda S}{\frac{K}{\lambda S}(\lambda - 1)\frac{s_1}{S}\frac{r}{E} + p\left[1 + \frac{V_t r}{2BS^2} + \frac{K}{\lambda S}(\lambda - 1)\frac{s_1}{S}\frac{m}{E}p + \frac{V_t m}{2BS^2}p^2\right]}$$

$V_t r/2BS^2$ is negligible compared with 1. Putting, as before,

$$\frac{K}{\lambda S} = \omega_f \quad , \quad \sqrt{\frac{2BS^2}{V_t m}} = \omega_c$$

and defining

$$\zeta_1'' = \frac{\omega_c}{2}\frac{K}{\lambda S}(\lambda - 1)\frac{s_1}{S}\frac{m}{E} \qquad \varepsilon_1'' = \frac{K}{\lambda S}(\lambda - 1)\frac{s_1}{S}\frac{r}{E} = \frac{2}{\omega_c}\frac{r}{m}\zeta_1''$$

we have

$$\boxed{H_1 = \frac{z}{x - z} = \frac{\omega_f}{\varepsilon_1'' + p\left(1 + 2\zeta_1''\frac{p}{\omega_c} + \frac{p^2}{\omega_c^2}\right)}} \tag{57}$$

The pressure feedback system introduces the same static position error as the various corresponding leakage processes for stabilization. This can be seen by simply replacing A or q_0/P_1 by $\omega_f(\lambda - 1)s_1/SE$.

Detected pressure transformed to a force applied to the valve spool (see *Figure 7.17*)—The application of a *force* to the valve will obviously have no effect if the main input is its own *position*. A non-rigid element must therefore be introduced between the input (position x) and the valve (joint α). Let

x = position of input component
x' = position of point α of the control rod
C = stiffness of contrlol inkage
s_2 = area of the detection piston.

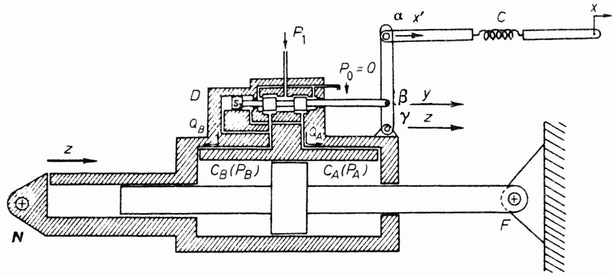


Figure 7.17

We will assume that the valve is balanced, i.e. that it has no hydraulic stiffness.

The open-loop transfer function is

$$\frac{z}{x-z} = \frac{z}{x'-z}\,\frac{x'-z}{x-z} = \frac{K/\lambda S}{p\left(1+\dfrac{V_t m}{2\,BS^2}p^2\right)}\,\frac{x'-z}{x-z}$$

if we neglect friction and since $V_t r/2BS^2$ is negligibly small compared with unity.

For equilibrium of the control lever, $\alpha\beta\gamma$ (moments about γ equal zero), we have

$$(P_A - P_B)s_2 = \lambda C\,(x - x')$$

Finally,

$$H_1 = \frac{z}{x-z} = \frac{K/\lambda S}{\dfrac{K}{\lambda S}\,\dfrac{1}{\lambda}\,\dfrac{s_2}{S}\,\dfrac{r}{C} + p\left(1+\dfrac{K}{\lambda S}\,\dfrac{1}{\lambda}\,\dfrac{s_2}{S}\,\dfrac{m}{C}\,p + \dfrac{V_t m}{2\,BS^2}p^2\right)}$$

Defining

$$\zeta_1''' = \frac{\omega_c}{2}\,\frac{K}{\lambda^2 S^2}\,\frac{s_2 m}{C} \qquad \varepsilon_1''' = \frac{K}{\lambda^2 S^2}\,\frac{s_2 r}{C} = \frac{2}{\omega_c}\,\frac{r}{m}\,\zeta_1'''$$

we can write

$$\boxed{H_1 = \frac{z}{x - z} = \frac{\omega_f}{\varepsilon_1''' + p\left(1 + 2\,\zeta_1''' \dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right)}} \tag{58}$$

This result is the same as that obtained in the previous Section. For equivalent systems

$$A = \frac{q_0}{P_1} = \frac{K}{\lambda S}(\lambda - 1)\frac{s_1}{S}\frac{1}{E} = \frac{K}{\lambda^2 S^2}\frac{s_2}{C}$$

the static error in position is the same as that introduced by the leakage stabilization processes.

Comparison—The results are summarized in *Table 7.3.*

Table 7.3

	Leakage flow in the		*Pressure feedback as*	
	ram	*valve*	*position*	*force*
Damping term, $2\,\zeta_1/\omega_c$	$\frac{m}{S^2}A$	$\frac{m}{S^2}\frac{q_0}{P_1}$	$\frac{m}{S^2}\frac{Ks_1}{E}\frac{(\lambda-1)}{\lambda}$	$\frac{m}{S^2}\frac{Ks_2}{C}\frac{1}{\lambda^2}$
Constant term, ε_1	$\frac{r}{S^2}A$	$\frac{r}{S^2}\frac{q_0}{P_1}$	$\frac{r}{S^2}\frac{Ks_1}{E}\frac{(\lambda-1)}{\lambda}$	$\frac{r}{S^2}\frac{Ks_2}{C}\frac{1}{\lambda^2}$

We have therefore shown that, as far as stability, accuracy and response speed are concerned, *leakage flow in the ram, leakage flow in the valve and pressure feedback are exactly equivalent methods of stabilization.*

There are, however, certain processes which can be used to eliminate the static error. In Appendix 7.1, for example, we will consider stabilization by dynamic pressure feedback.

We could say that *leakage flow* in the ram or valve is simply a *particular method of pressure detection.* Nevertheless, there are considerable differences between these direct and indirect methods of pressure detection.

We have already made a comparison between the values of the leakage flows in the ram and the valve.

As far as true pressure feedback is concerned, we should note that the introduction of the detected pressure in the form of a force at the control valve stage, a force which is therefore transmitted to the pilot's hand, *causes the loss of reversibility of the servocontrol.* This fact is very important. It is considered by aircraft pilots sometimes as just interesting, since it automatically causes an artificial feel, and sometimes as catastrophic, since it can cause the onset and amplification of flutter. In addition, the force necessary for stabilization can be much too large for an admissible artificial feel.

It has been assumed implicitly that the secondary feedback necessary for stabilization is a linear function of pressure, which is reasonable when the operation of the servocontrol is expressed by the linear equations of Section 7.4.2. In fact, however, these are only an approximation which becomes less and less valid as the load approaches its maximum value. The valve gain decreases with the loss of head in the valve as the pressure difference between the two chambers increases (cf. Section 7.6). We can see, therefore, that a detection which is an increasing function of $(P_A - P_B)$ but with a decreasing derivative, would be sufficient.

Without entering into a complicated analysis of this problem, we will simply state that this effect enables the maximum static error to be appreciably reduced and also that it often simplifies the introduction of secondary detection.

7.7.2.4. *Acceleration feedback*

By putting $u = 2$ in the general equation (54) and omitting the $(P_A - P_B)$ feedback terms, we obtain the transfer function

$$\frac{z}{x - z} = \frac{K/\lambda S}{p\left(1 + \frac{V_t r}{2\,BS^2} + \frac{K}{\lambda S}\beta p + \frac{V_t m}{2\,BS^2} p^2\right)}$$

Neglecting $V_t r / 2BS^2$ and defining

$$\zeta_2 = \frac{\omega_c}{2}\,\frac{K}{\lambda S}\,\beta$$

we get

$$\boxed{H_1 = \frac{z}{x - z} = \frac{\omega_f}{p\left(1 + 2\,\zeta_2 \frac{p}{\omega_c} + \frac{p^2}{\omega_c^2}\right)}} \qquad (59)$$

Acceleration feedback does not introduce a position error. Since its practical construction is difficult, it is used only when static accuracy is of prime importance. A description of acceleration detectors will not be given here.

However, it should be noted that some authors commend indirect detection of acceleration which is absolutely wrong. Their reasoning is as follows: 'Since the flow is roughly proportional to the speed of the ram, all that is necessary to obtain the acceleration is to detect and differentiate the flow'. Now this argument loses its validity, i.e. the speed ceases to be proportional to the flow, exactly at the frequencies for which we require stabilization. A simple calculation demonstrates the flaw in the reasoning mentioned. Suppose we have detected the derivative of the flow* or even some function of the flow, $z_D = Q . F(p)$ and that we have reintroduced it at the x stage (or x'):

$$Q = \frac{K}{\lambda}\,(x - z - z_D)$$

$$z_D = Q \quad F(p) = \frac{K}{\lambda}\,F(p)\,(x - z - z_D)$$

* See foot of page 273

From this, it can be shown that

$$z_D = \frac{KF(p)/\lambda}{1 + KF(p)/\lambda}(x - z)$$

so that the flow equation becomes

$$\frac{K}{\lambda}(x - z - z_D) = \frac{K}{\lambda}\left(1 - \frac{KF(p)/\lambda}{1 + KF(p)/\lambda}\right)(x - z)$$

$$= \frac{K/\lambda}{1 + KF(p)/\lambda}(x - z) = Spz + \frac{V_t}{2B}(P_A - P_B)p$$

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* *Detection of the derivative of flow*

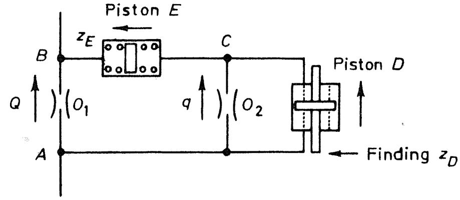


Figure 7.18 Flow coefficients, $a_{1,2}$; stiffness of springs, $R_{D,E}$; areas of pistons, $S_{D,E}$; run of pistons, $z_{D,E}$.

Flow through O_1: $\quad Q = \alpha_1(P_A - P_B) \quad (a)$

Flow through O_2: $\quad q = \alpha_2(P_A - P_C) \quad (b)$

Equilibrium of E: $\quad P_C - P_B = \dfrac{R_E z_E}{S_E} \quad (c)$

Equilibrium of D: $\quad P -_A P_C = \dfrac{R_D z_D}{S_D} \quad (d)$

Continuity of flow at junction C:

$$q = (z_E S_E - z_D S_D)p \qquad (e)$$

From (b) and (d) we get

$$q = a_2(P_A - P_C) = \frac{a_2 R_D z_D}{S_D} = (z_E S_E - z_D S_D)p$$

$$z_E = z_D\left(\frac{S_D}{S_E} + \frac{a_2 R_D}{p S_E S_D}\right) \qquad (f)$$

From $(c) + (d)$ and (a) we get

$$\frac{R_E z_E}{S_E} + \frac{R_D z_D}{S_D} = \frac{Q}{a_1}, \qquad z_E = \frac{Q}{a_1}\frac{S_E}{R_E} - z_D\frac{R_D}{R_E}\frac{S_E}{S_D} \qquad (g)$$

wherefrom, equating (f) and (g)

$$z_D\left[\frac{R_E}{S_E^2} + \frac{R_D}{S_D^2} + \frac{a_2}{p}\frac{R_E R_D}{S_E^2 S_D^2}\right] = \frac{Q}{a_1 S_D}$$

Putting $\quad \dfrac{S_E^2}{R_E} = \sigma_E \qquad \dfrac{S_D^2}{R_D} = \sigma_D$

we have

$$\frac{z_D}{Q} = \frac{\sigma_E \sigma_D}{a_1 a_2 S_D}\,\frac{p}{1 + \dfrac{\sigma_E + \sigma_D}{a_2}p}$$

which is effectively equivalent to a differentiation for $\omega \ll \alpha_2/(\sigma_E + \sigma_D)$.

The analysis now takes the same form as the general analysis, except that K/λ, and likewise the final transfer function containing K/λ, is multiplied by

$$\frac{1}{1 + \frac{K}{\lambda} F(p)}$$

There is no damping term introduced. The aim of the method is not achieved; the only favourable result is a certain filtering of high frequencies, exactly as in the case of using a dash-pot, which will now be examined.

7.7.3. STABILIZATION BY DASH-POT (*Figure 7.19*)

A dash-pot mounted between the spool and the body of the valve is quite often used as a stabilizing device in servocontrols.

As in the case of pressure feedback in the form of a *force* (Section 7.7.2.2) it is obvious that this process, which is effected by the application of a certain force

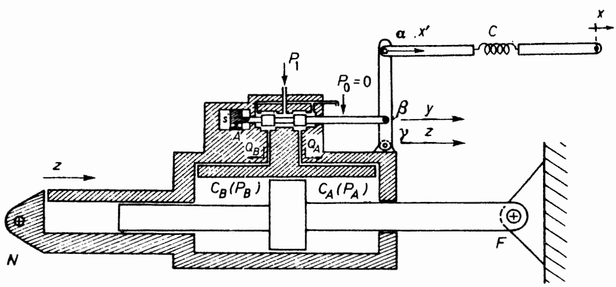


Figure 7.19

to the valve spool, will be ineffective if the input is the *position* of point α of the control lever rod. We therefore examine the action of the dash-pot only in the case where the input (position x) is connected to α by a flexible linkage system. Let

C = stiffness of the control linkage system
s' = area of the dash-pot piston
V' = half volume of the dash-pot
$\Delta P'$ = pressure difference between the two dash-pot chambers
q' = flow in the dash-pot
A' = dash-pot flow coefficient ($A' = q'/\Delta P'$)

Determination of the transfer function

$$H_1 = \frac{z}{x - z} = \frac{z}{x' - z} \frac{x' - z}{x - z}$$

Neglecting friction (and $V_t r/2BS^2$ in comparison with 1)

$$\frac{z}{x' - z} = \frac{\omega_f}{p\,(1 + p^2/\omega_c^2)}$$

Determination of $(x'-z)/(x-z)$

Equilibrium of control lever rod: $\Delta P's' = \lambda C(x-x')$

Dash-pot flow equation: $s'(y-z)p = \Delta P'(A'+pV'/2B)$

Kinematic equation: $x'-z = \lambda(y-z)$

Eliminating $\Delta P'$ and y gives

$$\frac{x'-z}{x-z} = \frac{1+pV'/2BA'}{1+\left(\dfrac{s'^2}{\lambda^2 CA'}+\dfrac{V'}{2BA'}\right)p}$$

wherefrom

$$\boxed{H_1 = \frac{z}{x-z} = \frac{\omega_f}{p\left(1+\dfrac{p^2}{\omega_c^2}\right)} \; \frac{(1+pV'/2BA')}{\left[1+\left(\dfrac{s'^2}{\lambda^2 CA'}+\dfrac{V'}{2BA'}\right)p\right]}} \qquad (60)$$

The dash-pot does not introduce a damping term into the transfer function, it only multiplies it by a lag term of the form $(1+tp)(1+Tp)$.

It would not be capable of stabilizing a servocontrol which had no natural damping, since it could not prevent infinite amplitude rise at $\omega = \omega_c$.

Its action is illustrated by the Nichols curves shown in *Figure 7.20*, which have been plotted neglecting $V'/2BA'$, i.e. t, which is very much smaller than T in practice and which has an effect only at very high frequencies. We can see that the effect of the dash-pot is

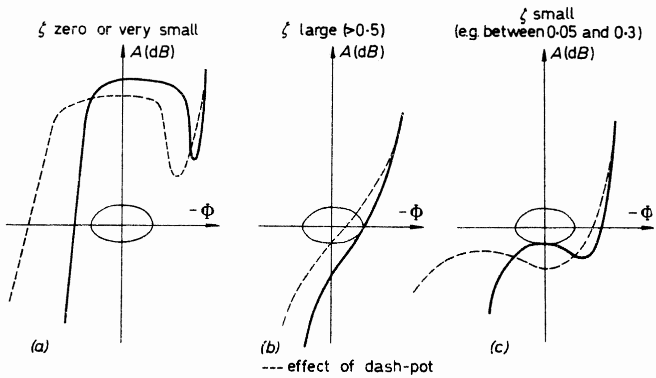


Figure 7.20

zero if the natural damping, ζ, is zero or extremely low (*Figure 7.20a*);

unfavourable if the natural damping is high, i.e. $> 0{\cdot}5$ (*Figure 7.20b*);

favourable if the natural damping is low, e.g. $0{\cdot}05 < \zeta < 0{\cdot}3$, which is usually the case (*Figure 7.20c*).

Note 1—We have assumed that the control valve has zero or negligible hydraulic stiffness. If this is not so, then, calling this hydraulic stiffness D, it can be shown that

$$\frac{x' - z}{x - z} = \frac{1}{1 + \dfrac{D}{C\lambda^2}} \quad \frac{1 + pV'/2BA'}{1 + \left[\dfrac{V'}{2BA'} + \dfrac{s'^2}{A'(\lambda^2 C + D)}\right] p} \tag{61}$$

The qualitative results remain unchanged.

Note 2—The dash-pot introduces a reaction at the pilot's manual control lever and can inconvenience him.

Its action is roughly proportional to the derivative of $(y-z)$. Since at low frequencies z can be considered as the integral of $(y-z)$ [in fact: $S\mathrm{d}z/\mathrm{d}t = K(y-z)$], the action of the dash-pot will be proportional to the second derivative of z.

For this reason, some writers define an 'equivalent mass' of the dash-pot, i.e. a mass giving the same reactions at the manual control lever. This mass is determined by comparing the equilibrium equations of the rod $\alpha\beta\gamma$ first with the dash-pot included and then just with the mass attached.

Its value is found to be

$$\mu = \frac{s_2'}{A' \lambda^2 \omega_f} \tag{62}$$

but it should be noted that this only applies at low frequencies ($\omega < \omega_f$).

7.8. EFFECT OF EXTERNAL PARAMETERS ON THE SERVOCONTROL

Sometimes the above calculations do not agree exactly with the actual behaviour of servocontrols, particularly with the large ones used in aircraft.

We must therefore take into account certain parameters which were formerly neglected. These are essentially

the elasticity of attachment of the servocontrol to the structure;
the elasticity of the connection between the ram and the load;
the elasticity of the control linkage.

We will examine their importance and determine the magnitude of their effects on the performance of the servocontrol (or, in the case of a proposed

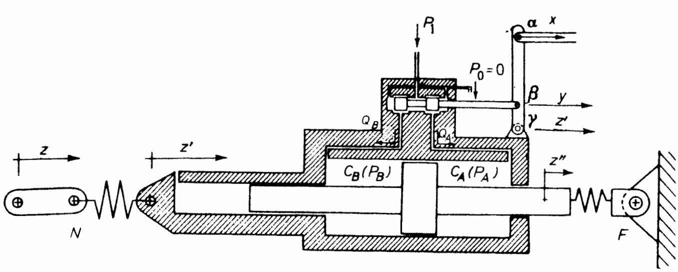


Figure 7.21

design, the tolerances to be imposed in order to avoid or limit operational troubles).

7.8.1. ELASTICITY OF ATTACHMENT BETWEEN RAM AND STRUCTURE AND BETWEEN RAM AND LOAD

7.8.1.1. *Deriving the equation*

Notation (see *Figure 7.21*)

z = position of load

z' = position of movable part of the ram ($z = z'$ if the connection between ram and load is rigid)

z'' = position of the fixed part of the ram (which is not really fixed unless the attachment to the structure is rigid)

N = stiffness of the connection between ram and load

F = stiffness of attachment between ram and structure.

(We will neglect friction forces in establishing the transfer function.)

Flow from the control valve

$$Q_A = Ke = K(y - z') = \frac{K}{\lambda}(x - z')$$

Volume of the chambers of the ram

$$\begin{cases} V_A = V_t + S(z' - z'') \\ V_B = V_t - S(z' - z'') \end{cases}$$

Flow equation:

$$\frac{K}{\lambda}(x - z') = (z' - z'')\,pS + \frac{V_t}{2B}\,p(P_A - P_B)$$

which can be written

$$\frac{K}{\lambda}(x - z) = Spz + \frac{V_t}{2B}\,p(P_A - P_B) + (z' - z)\left(\frac{K}{\lambda} + pS\right) - z''pS$$

Equilibrium of the load

$$N(z' - z) = z(r + mp^2)$$

Equilibrium of the 'fixed' part of the ram

$$(P_A - P_B)\,S = -Fz''$$

Equilibrium of the movable part of the ram

$$(P_A - P_B)\,S = N(z' - z)$$

Elimination of

$$z' - z = \frac{z}{N}(r + mp^2)\,; \quad (P_A - P_B) = \frac{N}{S}(z' - z) = \frac{z}{S}(r + mp^2)$$

and

$$z'' = -\frac{S}{F}(P_A - P_B) = -\frac{z}{F}(r + m p^2)$$

gives the transfer function

$$H_1 = \frac{z}{x - z}$$

$$= \frac{K/\lambda S}{\frac{K}{\lambda S}\frac{r}{N} + p\left[1 + \frac{V_t r}{2BS^2} + \frac{r}{N} + \frac{r}{F} + \frac{K}{\lambda S}\frac{m}{N}p + \left(\frac{V_t m}{2BS^2} + \frac{m}{N} + \frac{m}{F}\right)p^2\right]}$$

Thus, introducing the two quantities defined above, hydraulic stiffness, $r_h = 2BS^2/V_t$ and $\omega_f = K/\lambda S$ (Sections 7.5.1 and 7.5.2)

$$H_1 = \frac{z}{x - z} = \frac{\omega_f}{\omega_f \frac{r}{N} + p\left[1 + \frac{r}{r_h} + \frac{r}{N} + \frac{r}{F} + \omega_f \frac{m}{N}p + \left(\frac{m}{r_h} + \frac{m}{N} + \frac{m}{F}\right)p^2\right]}$$

As indicated before, r/r_h is negligible compared with 1. Also, however poorly the servocontrol is attached, the stiffnesses N and F are always much greater than r, so that in practice r/N and r/F can be neglected in comparison with 1. The result becomes

$$\boxed{H_1 = \frac{z}{x - z} = \frac{\omega_f}{\omega_f \frac{r}{N} + p\left[1 + \omega_f \frac{m}{N}p + \left(\frac{m}{r_h} + \frac{m}{N} + \frac{m}{F}\right)p^2\right]}} \qquad (63)$$

7.8.1.2. *Effect of elasticity of the attachment to the structure*

The attachment stiffness, F, does not appear in eqn. (63) except in the p^3 term

$$\frac{m}{r_h} = \frac{1}{\omega_c^2} \quad \text{becomes} \quad \frac{m}{r_h} + \frac{m}{F} = \frac{1}{\omega_c^2}\left(1 + \frac{r_h}{F}\right)$$

An attachment which is too elastic (stiffness F) lowers the resonant frequency by the factor $\sqrt{1/(1 + r_h/F)}$.

We could say that an increase in the elasticity of the attachment is equivalent to a decrease in the bulk modulus, B, of the fluid. We can therefore define an equivalent bulk modulus, B', by the equation:

$$B' = B\frac{1}{1 + r_h/F}$$

The disadvantages which are caused when ω_c is lowered have been discussed already. We must therefore find the minimum attachment stiffness necessary to ensure that the reduction in ω_c is less than n per cent. We have

$$\sqrt{\frac{1}{1 + r_h/F}} \geqslant 1 - \frac{n}{100}$$

so that

$$\frac{F_{\min}}{r_h} \geqslant \frac{100}{2\,n}$$

Calling the maximum load divided by half the maximum displacement the mean stiffness of the load ($r_m = F_R/z_M$), we know (Section 7.5.2) that

$$\frac{r_h}{r_m} = \frac{2\,B}{P_1}\,\frac{k_s}{k_v}$$

wherefrom

$$\boxed{F_{\min} \geqslant \frac{100}{n}\,\frac{B}{P_1}\,\frac{k_s}{k_v}\,r_m} \tag{64}$$

This equation gives the minimum acceptable value of the stiffness of a servocontrol attachment. For example, to limit the reduction of ω_c to 10 per cent in a ram where $P_1 = 200$ kg/cm^2, $B = 15{,}000$ kg/cm^2, $k_s = k_v$, we have

$$\frac{F_{\min}}{r_m} \geqslant 750$$

It is to be noted that this calculation does not demonstrate the stabilizing effect of the elasticity of attachment required by some manufacturers and often called '*structural feedback*'.

The reason is that this effect exists only in the presence of a particular kinematic arrangement of the control linkage. This is the particular case of the servo considered in the following Section, Note 2 (*Figure 7.22*) and of that with a fixed body and movable piston referred to in Section 7.9.1 (*Figure 7.26*).

7.8.1.3. *Effect of the elasticity of the connection between ram and load*

The connecting stiffness, N, has a more complex effect than the attachment stiffness, F. In eqn. (63) it appears in the p^3 term (reduction of ω_c), in the p^2 term (damping) and in a constant term (static error).

The p^3 term, mp^3/N, is completely analagous to the term mp^3/F introduced by the elasticity of the attachment, and the preceding Section can be repeated exactly with F replaced by N.

The p^2 term, $\omega_f mp^2/N$, introduces a reduced damping coefficient

$$\zeta = \omega_f \frac{m}{N}\,\frac{\omega_c'}{2}$$

where ω_c' is the new critical angular velocity

$$\frac{1}{\omega_c'^2} = \frac{m}{r_h} + \frac{m}{N}$$

This coefficient can be compared with the coefficient defined in Section 7.5.3, $\zeta_n = \omega_f/2\omega_c$, the damping coefficient necessary for the absolute condition of stability.

For the damping introduced by the elasticity of the connection to be sufficient for stability, we require $\zeta \geqslant \zeta_n$, i.e.

$$\frac{m}{N}\,\frac{\omega_c}{2} \geqslant \frac{1}{2\,\omega_c'} \qquad \text{or} \qquad N \leqslant m\,\omega_c'^2$$

which gives

$$\frac{N}{r_h} \leqslant 0$$

This condition is obviously impossible. *The damping due to the elasticity of the connection is never by itself sufficient to stabilize the servocontrol.*

A reduction in N causes a reduction of ω_c, so that it should not even be used to improve the stability of a servocontrol.

The constant term, $\omega_f r/N$, suppresses the open-loop integration. The open-loop gain becomes N/r.

The corresponding static error in the closed loop, r/N, is small since N is usually much greater than r.

It should be noted that the ratio of the constant term to the damping term is r/m. This result is the same as obtained for the stabilization processes listed in *Table 7.3.*

Note 1—The above analysis applies to a servocontrol with mechanical control in which the detection of the feedback quantity (the position of the body of the control valve) must be made at the level of the ram.

Now, in an electrohydraulic servocontrol, for example, the detector (a potentiometer) can be located either on the ram or the load without affecting the system.

We will consider the relative merits of the two positions.

Detector on the ram: The equations are exactly similar to those derived for the mechanical servocontrol and yield the same results.

Detector on the load: the flow from the control valve is given by

$$Q_A = Ke = K\,(y - z) = \frac{K}{\lambda}\,(x - z)$$

so that the flow equation is

$$\frac{K}{\lambda}\,(x - z) = (z' - z'')\,p\,S + \frac{V_t}{2\,B}\,p\,(P_A - P_B)$$

which can be written

$$\frac{K}{\lambda}\,(x - z) = S\,p\,z + \frac{V_t}{2\,B}\,p\,(P_A - P_B) + (z' - z)\,p\,S - z''\,p\,S$$

Finally, with the same approximations as before

$$\boxed{H_1 = \frac{z}{x - z} = \frac{\omega_f}{p\left[1 + \left(\frac{m}{r_h} + \frac{m}{F} + \frac{m}{N}\right)p^2\right]}} \tag{65}$$

This is to say, locating the detector directly on the load eliminates the position error and the damping term and conserves the p^3 term giving a reduction in ω_c.

Note 2—There are certain devices which reduce the disadvantages of elasticity of the attachment to the structure by providing a damping term similar to that which is introduced naturally by the elasticity of the connection to the load. We will briefly examine the device shown in *Figure 7.22.*

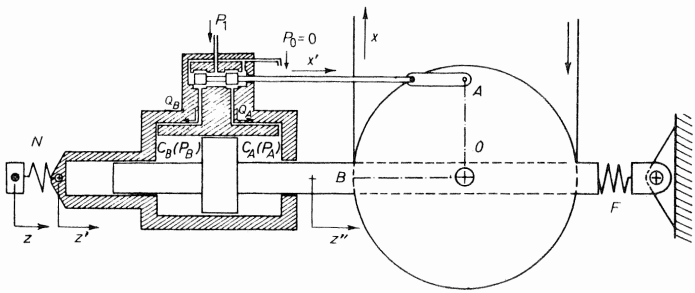


Figure 7.22

The control cable, which is perpendicular to the axis of the servocontrol, causes the rotation of a drum whose axis is attached to the 'fixed' part of the ram (position z''). The rotation of the drum controls the opening of the control valve by means of a connecting rod.

It can be seen from the figure that the device operates according to the equation

$$x' = \lambda x + z'' \qquad \text{where} \qquad \lambda = OA/OB$$

Since

$$Q_A = K(x' - z') = K[\lambda x - (z' - z'')] = Sp(z' - z'') + \frac{V_t}{2B}p(P_A - P_B)$$

the transfer function can be shown, in a manner similar to that used previously, to be

$$\frac{z}{\lambda x - z}$$

$$= \frac{K/S}{\frac{K}{S}\left(\frac{r}{N} + \frac{r}{F}\right) + p\left[1 + \frac{r}{N} + \frac{r}{F} + \frac{r}{r_h} + p\frac{K}{S}\left(\frac{m}{N} + \frac{m}{F}\right) + p^2\left(\frac{m}{N} + \frac{m}{F} + \frac{m}{r_h}\right)\right]}$$

The elasticities F and N now have absolutely equal effects.

A new damping term is introduced, $(K/S)(m/F)p^2$ (and also a static error term).

If the attachment of a servocontrol cannot be made more rigid, this process enables us to improve the stability, but at the cost of a deterioration in accuracy. (Section 7.9.1 describes another kinematic device which gives a similar effect.)

7.8.2. ELASTICITY OF THE CONTROL LINKAGE (*Figure 7.23*)

Elasticity of the control linkage will have an effect only if there is a tension or compression force in the linkage, i.e. if there are reactions at the valve or if the mass of the linkage itself introduces significant inertia forces.

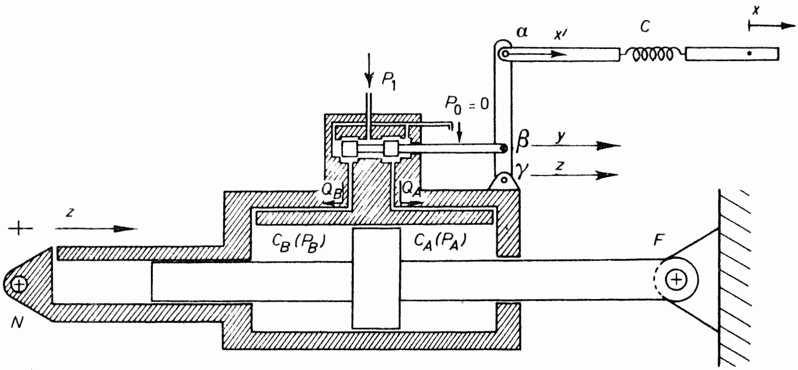


Figure 7.23

There are two cases to be considered:

(1) *when the forces applied to the linkage are a function of the absolute positions of different points of the linkage* (*Figure 7.24a*), i.e. where the linkage either has non-negligible mass or is connected to a fixed point by means of a spring, or where there is artificial 'feel'.

In this case, the transfer function x'/x is independent of the servocontrol and can be considered separately

$$\frac{z}{x} = \left[\text{closed loop of} \quad \frac{z}{x' - z} \right] \times \frac{x'}{x} \tag{66}$$

(2) *when the forces applied to the linkage are a function of certain internal parameters of the servocontrol* (*Figure 7.24b*), i.e. where there is

a hydraulic stiffness in the valve: force proportional to $(y-z)$;

a dash-pot attached to the valve: force proportional to $p(y-z)$;

a secondary pressure feedback in the form of a force: force proportional to $(P_A - P_B)$, etc.

In this case it is preferable to consider an artificial open loop (*Figure 7.24c*)

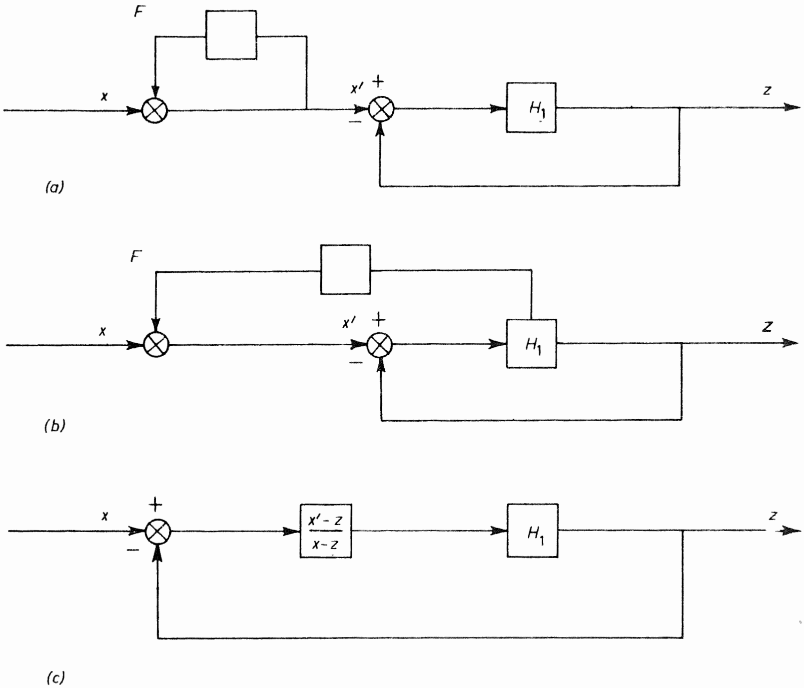


Figure 7.24

$$\frac{z}{x - z} = \frac{z}{x' - z} \; \frac{x' - z}{x - z}$$

$$\frac{\mathbf{z}}{\mathbf{x}} = \text{closed loop of} \quad \left[\frac{\mathbf{z}}{\mathbf{x}' - \mathbf{z}} \times \frac{\mathbf{x}' - \mathbf{z}}{\mathbf{x} - \mathbf{z}}\right] \tag{67}$$

The first case is entirely mechanical and is therefore outside the scope of this book.

It should be noted, however, that the mechanical analysis is straightforward if the constants of the system, such as mass and stiffness, can be considered as acting at a point but becomes more complicated if the distribution of mass, etc. has to be taken into account.

Inconvenient resonant phenomena can be encountered in large aircraft equipped with long linkage systems where the automatic pilot which operates the linkage is situated far upstream of the aerodynamic control surface.

We will now consider briefly the orders of magnitude involved.

Speed of propagation of motion in a linkage—If

μ = mass per unit length,
c = stiffness per unit length,
u = position of the point considered,

the differential equation of equilibrium of an elemental length, dl, of the linkage is

$$\mu \, \mathrm{d}l \frac{\partial^2 u}{\partial t^2} = c \, \mathrm{d}l \frac{\partial^2 u}{\partial l^2}$$

With $v = \sqrt{c/\mu}$ (= speed of propagation) this gives, for the general case

$$u = f\left(t - \frac{l}{v}\right) + g\left(t + \frac{l}{v}\right)$$

If the linkage is a uniform steel bar of density ρ and Young's modulus

$$E = \frac{F/s}{\Delta l/l}$$

then

$$\mu = s\varrho \qquad c = \frac{F}{\Delta l/l} = Es$$

so that

$$v = \sqrt{\frac{E}{\varrho}}$$

Thus, with $E = 20{,}000 \text{ kg/mm}^2$

and
$$\rho = \frac{w}{g} = \frac{8 \times 10^{-6} \text{ kg/mm}^3}{10^4 \text{ mm/sec}^2} = 8 \times 10^{-10} \frac{\text{kg/sec}^2}{\text{mm}^4}$$

$$v = 5{,}000 \text{ m/sec}$$

But a linkage is never uniform. The weight of supporting brackets, connecting levers, etc. increases the overall weight without increasing the stiffness. The speed of propagation will thus be less than 5,000 m/sec (*very* much less for a cable). For example, in a linkage whose weight is four times that of the uniform steel rod of the same elasticity, $v = 2{,}500$ m/sec.

Assuming that perturbation phenomena appear when the wavelength decreases to about 10 times the length of the linkage, the frequency should be limited to values less than $f = v/10L$. Thus, if $L = 35$ m (typical of large aeroplanes),

$$f = 7 \text{ c/s}$$

If the control is made by cables, this value is considerably reduced.

The analysis of the second case does not present any particular theoretical difficulties.

Three examples were considered earlier: (*a*) secondary pressure feedback in the form of a force applied to the control valve [*Figure 7.17*, eqn. (58)]; (*b*) stabilization by dash-pot [*Figure 7.19*, eqn. (60)]; (*c*) hydraulic stiffness of the valve.

[*Note 1* of Section 7.7.3, eqn. (61) reduces to

$$\frac{x' - z}{x - z} = \frac{1}{1 + D/C\lambda^2}$$

when there is no dash-pot.]

There does not seem to be much point in finding the solution for the general case:

$$[\text{force } a(y - z)\,p^{\alpha} + b\,z\,p^{\beta} + c\,(P_A - P_B)\,p^{\gamma} + \cdots]$$

7.9. BRIEF SURVEY OF DIFFERENT TYPES OF SERVOCONTROL

7.9.1. SERVOCONTROLS WITH GAIN $\neq 1$

We will not consider servocontrols of the type shown in *Figures 7.1* or *7.3* where a lever or some other system is installed on the linkage which has no common point with the movable part of the ram and serves only to introduce a constant multiplying term between command and output.

We will only deal with servocontrols where the control valve is attached to the fixed part of the ram and the connecting lever is hinged at a point on the movable part. This type of servocontrol is shown in *Figure 7.25* (note that the movable and fixed parts of the ram are interchanged compared with *Figures 7.3, 7.5, 7.12, 7.16, 7.17, 7.19, 7.21* and *7.22*).

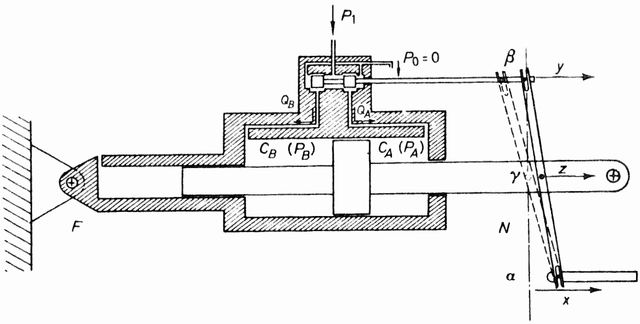


Figure 7.25

The servocontrol is in equilibrium when $y = 0$. We therefore have $z/x = \gamma\beta/\alpha\beta$; the connecting lever, $\alpha\beta\gamma$, is generally inclined at equilibrium (unless $z = x = 0$).

It is technically easier to supply fluid to a valve which is attached to a fixed component, but more difficult to design a connecting lever which is capable of large angular movement.

Theoretically, there is no fundamental difference between this system and the preceding ones. Using the same notation as above and replacing the kinematic ratio $\lambda = \overline{\alpha\gamma}/\overline{\beta\gamma}$ by the more convenient ratio $\mu = \overline{\alpha\beta}/\overline{\gamma\beta}$, we have

$$Q_A = K\,y = -\,S\,p\,z + \frac{V_t}{2\,B}\,p\,(P_A - P_B)$$

$$(P_A - P_B)\,S = -\,z\,(r + f\,p + m\,p^2)$$

and

$$(x - y) = \mu\,(z - y),$$

wherefrom the open-loop transfer function is found to be

$$\boxed{H_1 = \frac{\mu z}{x - \mu z} = \frac{\dfrac{K}{S}\dfrac{\mu}{\mu - 1}}{p\left[1 + \dfrac{V_t r}{2BS^2} + \dfrac{V_t f}{2BS^2}p + \dfrac{V_t m}{2BS^2}p^2\right]}} \tag{68}$$

This expression is similar to the transfer function of eqn. (6), with the exception that the output under steady conditions, which is equal to the input, is μz and not z.

Analysis of the case where the attachment to the structure and the connection between ram and load are not completely rigid—With the notation of Section 7.8.1.1 we have (*Figure 7.26*)

$$\begin{aligned}
Q_A &= Ke = K(y - z'') \\
V_A &= V_t - S(z' - z'') \\
V_B &= V_t + S(z' - z'') \\
Q_A &= -Sp(z' - z'') + \frac{V_t}{2B}p(P_A - P_B) \\
N(z' - z) &= -z(r + mp^2) \\
(P_A - P_B)S &= Fz'' \quad = N(z' - z)
\end{aligned}$$

wherefrom

$$H_1 = \frac{\mu z}{x - \mu z} = \frac{\dfrac{K}{S}\dfrac{\mu}{\mu - 1}}{\dfrac{K}{S}\dfrac{r}{F} + \dfrac{K}{S}\dfrac{\mu}{\mu - 1}\dfrac{r}{N} + p\left(1 + \dfrac{r}{N} + \dfrac{r}{F} + \dfrac{V_t r}{2BS^2}\right) + C}$$

where $C = p^2\left(\dfrac{K}{S}\dfrac{m}{F} + \dfrac{K}{S}\dfrac{\mu}{\mu - 1}\dfrac{m}{N}\right) + p^3\left(\dfrac{m}{F} + \dfrac{m}{N} + \dfrac{V_t m}{2BS^2}\right)$

For the reasons given in Section 7.8.1.1, r/N, r/F and $V_t r/2BS^2$ can be neglected.

Thus, introducing the hydraulic stiffness $r_h = 2BS^2/V_t$, we have

$$\boxed{H_1 = \frac{\mu z}{x - \mu z} = \frac{\dfrac{K}{S}\dfrac{\mu}{\mu - 1}}{\dfrac{K}{S}\dfrac{r}{F} + \dfrac{K}{S}\dfrac{\mu}{\mu - 1}\dfrac{r}{N} + p\left[1 + \left(\dfrac{K}{S}\dfrac{m}{F} + \dfrac{K}{S}\dfrac{\mu}{\mu - 1}\dfrac{m}{N}\right)p + \left(\dfrac{m}{N} + \dfrac{m}{F} + \dfrac{m}{r_h}\right)p^2\right]}} \tag{69}$$

This equation is very similar to eqn. (63), with the difference that, owing to the fact that the valve is now attached to the fixed part of the ram, the attachment stiffness is now added to the connection stiffness and introduces a static error and a damping term (both of which are, in any case, physically obvious).

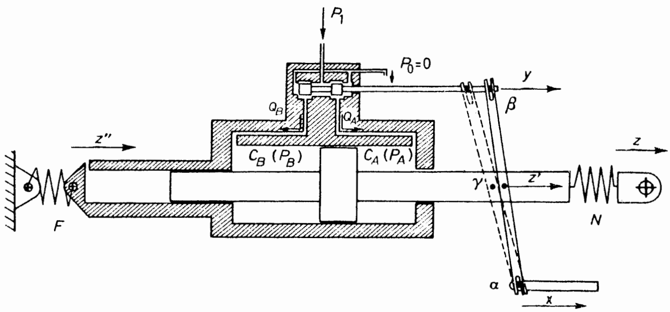


Figure 7.26

Some authors indicate that elasticity of attachment could be used to stabilize servocontrols. For the reasons given in Section 7.8.1.3, when dealing with the effect of elasticity of the connection on servocontrols with unit gain, we should not rely too heavily on such a possibility.

7.9.2. SERVOCONTROL WITH DIFFERENTIAL-AREA RAM

In a servocontrol with a differential-area ram, the valve controls just one of the two chambers (chamber B in *Figure 7.27*), the restoring force being provided either by a spring, by an opposing load or by the application of a constant pressure to the other chamber.

The most usual arrangement (*Figure 7.27*) is where the second chamber, of effective area half that of the active chamber, is under a constant pressure P_1. If there is no load, equilibrium is obtained when there is a pressure $P_1/2$ ($S \times P_1/2 = S/2 \times P_1$) in the active chamber.

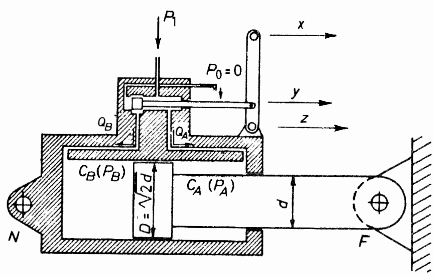


Figure 7.27

Comparison between differential and equal-area rams—This comparison will obviously be made between servocontrols of equal performance. The symbols relating to the differential-area ram will be primed ′, and S' will be taken as the area of the larger (active) chamber, that of the other being $S'/2$. With the notation used in Section 7.4.2, we have

$$x - z = \lambda (y - z)$$

$$Q'_B = - \frac{K'}{\lambda}(x - z) = - S' p z + \frac{V'_t}{B} P_B p$$

$$\frac{P_1 S'}{2} - P_B S' = z (r + f p + m p^2)$$

wherefrom, by eliminating P_B

$$\frac{z}{x - z} = \frac{K'/\lambda S'}{p\left(1 + \frac{V'_t r}{BS'^2} + \frac{V'_t f}{BS'^2} p + \frac{V'_t m}{BS'^2} p^2\right)} \tag{70}$$

A servocontrol with an equal-area ram of the same performance will have the transfer function (cf. Section 7.4.2)

$$\frac{z}{x - z} = \frac{K/\lambda S}{p\left(1 + \frac{V_t r}{2\,BS^2} + \frac{V_t f}{2\,BS^2} p + \frac{V_t m}{2\,BS^2} p^2\right)} \tag{6}$$

Table 7.4 shows that $K = K'/2$, $S = S'/2$ and $V_t = V'_t/2$. The two transfer functions are therefore identical.

	Differential area	*Equal area*	*Comparison*
Open-loop gain	$\omega'_f = \frac{K'}{\lambda S'} = \frac{K' P_1}{2\,\lambda F_M}$	$\omega_f = \frac{K}{\lambda S} = \frac{K P_1}{\lambda F_M}$	$\omega'_f = \omega_f$
Critical frequency	$\omega'_c = \sqrt{\frac{BS'^2}{V'_t m}} = \sqrt{\frac{2\,BF_M}{P_1 k_v m z_M}}$	$\omega_c = \sqrt{\frac{2\,BS^2}{V_t m}} = \sqrt{\frac{2\,BF_M}{P_1 k_v m z_M}}$	$\omega'_c = \omega_c$
Reduced damping coefficient	$\zeta' = \frac{\omega'_c}{2}\frac{V'_t f}{BS'^2} = \frac{f}{2\,m\omega'^2_c}$	$\zeta = \frac{\omega_c}{2}\frac{V_t f}{2\,BS^2} = \frac{f}{2\,m\omega^2_c}$	$\zeta' = \zeta$

It should be noted that, away from the middle position, the result is clearly favourable on one side (volume of the active chamber decreasing) and unfavourable on the other, while for the servocontrol with the equal-area ram the increase in volume of one chamber is almost compensated by the decrease in volume of the other.

The servocontrol with a differential-area ram will have the same *static* and *dynamic* performance as one with an equal-area ram, provided that it has

Table 7.4

	Differential area	*Equal area*	*Comparison*
Area necessary to produce force F_M	$S' = \dfrac{2F_M}{P_1}$	$S = \dfrac{F_M}{P_1}$	$S' = 2S$
Effective half volume	$V'_u = S' z_M = \dfrac{2F_M z_M}{P_1}$*	$V_u = S z_M = \dfrac{F_M z_M}{P_1}$	$V'_u = 2V_u$
Total half volume	$V'_t = k_v V'_u$*	$V_t = k_v V_u$	$V'_t = 2V_t$
Maximum instantaneous hydraulic flow necessary to the valve†	$Q'_M = S'\left(\dfrac{dz}{dt}\right)_M = \dfrac{2F_M}{P_1}\left(\dfrac{dz}{dt}\right)_M$	$Q_M = S\left(\dfrac{dz}{dt}\right)_M = \dfrac{F_M}{P_1}\left(\dfrac{dz}{dt}\right)_M$	$Q'_M = 2Q_M$
Mean hydraulic flow necessary to give N c/s of **amplitude z_M**	$Q'_m = 2S'Nz_M = \dfrac{4F_M}{P_1}Nz_M$	$Q_m = 4SNz_M = \dfrac{4F_M}{P_1}Nz_M$	$Q'_m = Q_m$
Valve gain necessary at the same opening	$K' = \dfrac{Q'_M}{e}$	$K = \dfrac{Q_M}{e}$	$K' = 2K$
Hydraulic stiffness	$r'_h = \dfrac{BS'^2}{V'_t}$	$r_h = \dfrac{2BS^2}{V_t}$	$r'_h = r_h$

* On one side only.

† The instantaneous supply flow from the valve is double during the filling of the active chamber and zero during the emptying, but the instantaneous flow measured in the supply pipe from P_1, upstream of the connection to the non-active chamber, is the same. What happens is that, during the filling of the active chamber, half the flow is supplied from the emptying of the non-active chamber, and during the emptying of the active chamber, flow has to be supplied to fill the non-active chamber.

the same travel, twice the effective area and twice the valve gain. *It has the advantages of a simpler control valve* (3 *way instead of* 4) *and a correspondingly simpler hydraulic circuit but the disadvantage of a larger size.*

7.9.3. SERVOCONTROL WITH ROTATIONAL MOTOR

Up to now we have considered servocontrols with rams which are effectively linear motors. Such a ram can be replaced by a rotational hydraulic motor, which can be of the type with gears (*Figure 7.28*), pistons or vanes. All the valve layouts previously dealt with can be retained.

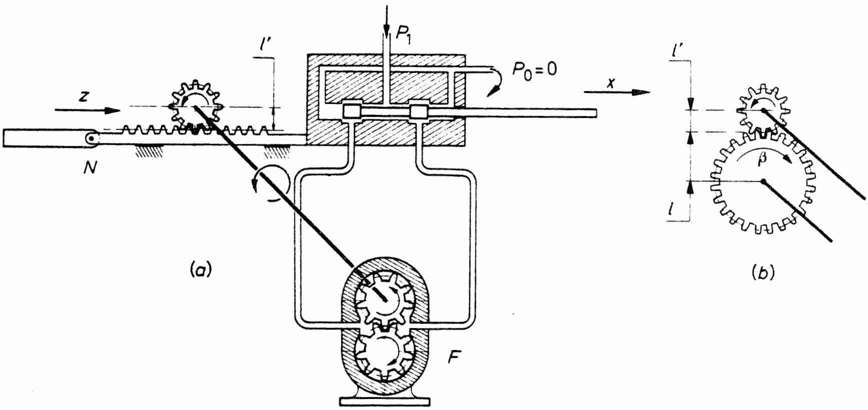


Figure 7.28. Servocontrol with rotational motor: (*a*) linear output, (*b*) rotational output

If the input and output were rotational, it would be technologically interesting to replace the linear valve by a rotational one.

Some characteristics of servocontrols with rotational motors

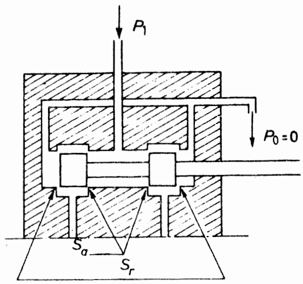
(*a*) *Leakage flow*—Although it is easy to make a linear actuator having no leakage between the two chambers, there is always a significant leakage flow in rotational motors. Since the leakage increases during the life of the motor and depends on the state of the liquid, it introduces a factor of uncertainty.

(*b*) *Irregularities of starting*—In a gear motor and especially in a piston motor, even when there is no load, an appreciable ΔP is required to set the motor in motion in one direction or the other, while immediately after it has started a much smaller ΔP suffices to maintain motion. The magnitude of ΔP required to start the motor also varies according to the geometric position occupied before starting. This phenomenon causes irregularities of operation and in certain cases a static inaccuracy.

It is possible to reduce this disadvantage considerably by the use of a special type of valve having permanent leakage flow, in which the restriction areas on the supply side are very small, s_a, and those on the sump side are very large, s_r, as shown in *Figure 7.29*.

In this case, the pressures in the two chambers are quite clearly less than $P_1/2$ when the valve is in equilibrium*. The hydraulic forces and, therefore, the reaction and friction forces on the bearings and pistons are considerably reduced and the value of ΔP required for starting is accordingly very much lower.

Figure 7.29



(*c*) *The resonance frequency*, ω_c—The resulting equation is identical with that of a servocontrol with a ram.

If V_m is the total capacity of the hydraulic motor and $v_m = V_m/2\pi$ the capacity per radian, then, if the motor is 100 per cent efficient, the torque produced under a pressure difference ΔP is

$$C = \Delta P\, v_m,$$

since after an angular displacement β the mechanical work done, $C\beta$, is equal to the work done by the liquid, $\Delta P v_m \beta$.

If l' is the pitch radius of the pinion engaging the rack (see *Figure 7.28*), *the hydraulic motor is equivalent to a linear actuator of effective area* $S = v_m/l'$. Eqn. (6), (7) and (8) for the transfer function and the expression for the resonant frequency, eqn. (10), are therefore the same, provided S is replaced by v_m/l'.

The two systems are not completely analagous, however. In the analysis of the linear actuator we have defined a volumetric coefficient characteristic of the actuator and almost fixed for any given size and performance specification

$$k_v = \frac{V_t}{S z_M}$$

together with a power coefficient

$$k_s = \frac{P_1 S}{F_R}$$

* The mean pressure in the two chambers is

$$P_m = P_1 \frac{1}{1 + (s_r/s_a)^2}$$

By equating the flows in the two restrictions we get

$$q = K s_a \sqrt{P_1 - P_m} = K s_r \sqrt{P_m}$$

The critical angular frequency was expressed in the form

$$\omega_c = \sqrt{\frac{2B}{P_1}\frac{k_s}{k_v}\frac{F_R}{mz_M}} \tag{12}$$

showing that, if P_1 and k_s are fixed, there is an upper limit to ω_c, since the other parameters in eqn. (12) cannot be varied.

In the case of a rotary actuator, the volumetric coefficient, which is important since it is characteristic of the actuator and is almost fixed for any given size and performance specification, is

$$k'_v = \frac{V_t}{v_m}$$

i.e. the ratio of the total volume, V_t, under pressure, P_A or P_B, to the displacement volume per radian, v_m.

With this definition of k'_v and defining as before k_s as the ratio of the motive force to the resisting force

$$k_s = \frac{P_1 v_m}{lF_R}$$

we get

$$\omega'_c = \sqrt{\frac{2B}{P_1}\frac{k_s}{k'_v}\frac{F_R}{ml'}} \quad \left(\text{or} \quad \omega'_c = \sqrt{\frac{2B}{P_1}\frac{k_s}{k'_v}\frac{C_R}{I}\frac{l}{l'}}\right) \tag{12'}$$

This is the same as eqn. (12) except that l' replaces z_M (or l'/l replaces β_M).

In theory, there is no upper limit to ω'_c since l' (or l'/l) can be made as small as required. There is a simple physical explanation of this difference. In the case of the linear actuator, the total volume, V_t, which appears in the expression for the natural frequency, is in fact fixed by the operational requirements. (If, for example, the driving lever arm, l, is multiplied by a factor λ, the effective area of the actuator, S, is divided by λ and its maximum displacement, z_M, is multiplied by λ, so that V_v, which is the product of Sz_M, remains unchanged while $V_t = k_v S z_M$ is changed only by any small effect on k_v due to the change in design.)

In the case of the rotary actuator, however, the volume V_t is approximately divided by λ when the scale of the motor is reduced by a factor λ. To produce the same force (or torque) at the output, the lever arm l' must also be divided by λ. The motor makes more revolutions for a given load displacement and rotates faster for a given load velocity.

Upper limit of ω'_c in practice—In a real system there are several reasons which prevent ω'_c being raised indefinitely:

(a) the reduction in motor size is not usually accompanied by a proportional reduction in dead volume (valves, tubing);

(*b*) the increase in gear ratio which should take place with the reduction in motor size introduces technical problems and eventually causes a loss of efficiency;
(*c*) the increase in motor speed which is associated with its decrease in size may exceed the limit imposed by its manufacturer;
(*d*) finally, the effect of the increase in speed on the inertia forces of the motor usually predominates over the effect of the reduction in size, so that the inertia of the motor may become very large in comparison with that of the load.

The relative merits of linear and rotary actuators—The comparison made here between linear and rotary motors is based solely on their respective dynamic performances, but it must be remembered that this is not the only nor necessarily even the most important consideration in the choice of motor.

It has been shown that the performance of the linear actuator is limited since, after making P_1 as small as possible, k_s as large as possible and k_v as small as possible, there is no other way of increasing ω_c.

This is not so for the rotary actuator.

In this case, the first step is to find the range of values of l' (or l'/l) in which the rotary actuator is likely to be more suitable.

The upper limit of l' (or l'/l)—A comparison of eqn. (12) and (12′) for ω_c and ω_c' shows that the condition $\omega_c' > \omega_c$ may be written

$$l' < \frac{k_v}{k_v'} z_M \qquad \left(\text{or} \quad \frac{l'}{l} < \frac{k_v}{k_v'} \beta_M\right)$$

It is therefore only necessary to know the values of k_v and k_v' to determine the upper limit of l' (or l'/l).

k_v, whose theoretical lower limit is 1, normally lies between 1·2 for large rams and 1·5 for small rams or those equipped with internal locking devices, but poor design can result in much higher values.

k_v', whose theoretical lower limit is $\pi/2$ (since $V_t \geqslant \frac{1}{4}.2\pi v_m$), is normally very much higher, especially when an existing motor and valve have been coupled together. By redesigning the motor and control valve housings, using solid pistons and taking many precautionary measures, it is possible to obtain a value of k_v' of 4 for a motor of $v_m = 1{\cdot}3\ \text{cm}^3$ ($= 8\ \text{cm}^3$ per revolution). But values of the order of 10 or even 20 are quite common.

Lower limit of l' (or l'/l)—Let $(\mathrm{d}\theta/\mathrm{d}t)_{\max}$ be the motor speed limit specified by the manufacturer and $(\mathrm{d}z/\mathrm{d}t)_{\max}$ [or $(\mathrm{d}\beta/\mathrm{d}t)_{\max}$] the maximum load speed required. The limiting value of l' is therefore

$$l' > \frac{(\mathrm{d}z/\mathrm{d}t)_{\max}}{(\mathrm{d}\theta/\mathrm{d}t)_{\max}} \qquad \left(\text{or} \quad l' > \frac{l(\mathrm{d}\beta/\mathrm{d}t)_{\max}}{(\mathrm{d}\theta/\mathrm{d}t)_{\max}}\right)$$

Optimization of l'—If the two limiting values of l' are close, or if the second is greater than the first, the rotary actuator is unsuitable.

If this is not the case, values of ω_c' must be calculated for several values of l' between the two limits; but this time the moment of inertia of the motor, I_m, must be taken into account by replacing the mass of the load, m, by $M+I_m/l'^2$, or its moment of inertia, I, by $I+I_m(l/l')^2$.

The upper limit of l' (or l'/l) is therefore given by

$$l' < \frac{k_v}{k_v'} \cdot \frac{z_M}{(1+I_m/ml'^2)} \qquad \left(\text{or} \quad \frac{l'}{l} < \frac{k_v}{k_v'} \frac{\beta_M}{(1+I_m l^2/Il'^2)}\right)$$

7.9.4. FORCE SERVOCONTROLS

The output from a servocontrol is sometimes required in the form of a *force* rather than a *position** (see Section 8.2).

The control of the position of the ram is therefore replaced by a control of the force, which is generally attained by detecting the pressure difference, $P_A - P_B$, between the two chambers. Strictly speaking, this method of detection is not correct, since it measures the sum of the external force, the friction force and the inertia force

$$[(P_A - P_B)\, S = z\,(r + f\,p + m\,p^2)]$$

but it is easy to achieve in practice (*Figure 7.30*) and it also has the advantage of introducing an appreciable damping term, as we will see later.

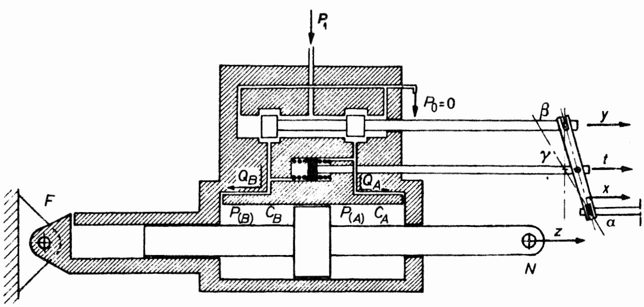


Figure 7.30

Deriving the equation—The ideal force servocontrol would have the equation $F_a = kx$ where F_a is the force resisting the ram and k is a constant.

A real force servocontrol has the closed-loop transfer function

$$\frac{F_a}{kx} = H(p)$$

* This can be useful in an aircraft or missile to counteract the reduction in aerodynamic control surface effectiveness with speed and altitude. A given signal corresponds to a constant hinge moment of the control surface, and the load factor is therefore effectively constant.

which can be investigated using the imaginary open loop

$$\frac{F_a}{kx - F_a}$$

The opposing force, F_a, is a function of z and of time.

But F_a often varies sufficiently slowly and regularly that, over short periods of time and for small amplitudes, we can write

$$F_a = rz$$

the stiffness, r, being *provisionally* and *locally* considered as constant.

This assumption will be made for the stability analysis. (The assumption is valid only if the frequency of variation of r is appreciably lower than the critical frequencies which occur in the analysis; this is usually the case.)

We therefore require

$$\frac{rz}{kx - rz}$$

Notation (see *Figure 7.30*)

y = valve opening
t = displacement of pressure difference detector
s = effective area of this detector
c = total stiffness of its springs
μ = the ratio $\alpha\beta/\gamma\beta$.

Under stable conditions, $y = 0$, $x = \mu t$

$$(P_A - P_B)\, s = ct$$

$$(P_A - P_B)\, S = F_a = rz$$

wherefrom

$$F_a = rz = x\frac{cS}{\mu s}$$

The constant k, defined above, is therefore equal to $cS/\mu s$. Under transient conditions, we have

kinematic equation: $$x - y = \mu\,(t - y) \tag{71}$$

flow equation (*Figure 7.30*):

$$Q_A = -Ky = Spz + \frac{V_t}{2B}p\,(P_A - P_B) \tag{72}$$

equilibrium of ram (neglecting friction):

$$(P_A - P_B)\, S = z\,(r + mp^2) \tag{73}$$

equation of detector (neglecting friction, inertia and compressibility of the oil in the detector)

$$t = (P_A - P_B)\frac{s}{c} \tag{74}$$

By the successive elimination of t, y and $(P_A - P_B)$ we get

$$x = \mu l + y(1-\mu) = \mu \frac{s}{c}(P_A - P_B) + y(1-\mu)$$

$$= z\frac{\mu - 1}{K} Sp + (P_A - P_B)\left(\mu\frac{s}{c} + \frac{\mu-1}{K}\frac{V_t}{2B}p\right)$$

$$= z\left[\frac{\mu-1}{K} Sp + \left(\frac{\mu}{S}\frac{s}{c} + \frac{\mu-1}{K}\frac{V_t}{2BS}p\right)(r + mp^2)\right]$$

Multiplying both sides by k and noting that $\mu s/Sc = 1/k$ gives

$$kx = kz\left[\frac{\mu-1}{K}Sp + \frac{r}{k} + \frac{m}{k}p^2 + \frac{\mu-1}{K}\frac{V_t r}{2BS}p + \frac{\mu-1}{K}\frac{V_t m}{2BS}p^3\right]$$

wherefrom

$$H_1 = \frac{rz}{kx - rz} = \frac{\dfrac{Kr}{k(\mu-1)S}}{p\left(1 + \dfrac{V_t r}{2BS^2} + \dfrac{Km}{k(\mu-1)S}p + \dfrac{V_t m}{2BS^2}p^2\right)} \tag{75}$$

Using the notation for hydraulic stiffness adopted previously

$$\frac{2BS^2}{V_t} = r_h$$

and the open-loop gain, here

$$\left[\omega_f = \frac{K_r}{k(\mu-1)S}\right]$$

eqn. (75) becomes

$$\boxed{H_1 = \frac{rz}{kx - rz} = \frac{\omega_f}{p\left(1 + \dfrac{r}{r_h} + \dfrac{m}{r}\omega_f p + \dfrac{m}{r_h}p^2\right)}} \tag{76}$$

Putting $\omega_f = \lambda\sqrt{r/m}$, this becomes (*Figure 7.31*)

$$H_1 = \frac{1}{\dfrac{p}{\omega_f}\left(1 + \dfrac{r}{r_h} + \dfrac{\lambda^2 p}{\omega_f} + \dfrac{\lambda^2 r}{r_h}\dfrac{p^2}{\omega_f^2}\right)}$$

(*a*) *Theoretical stability*—If r is no longer neglected in comparison with r_h, in order to deal with irregular values of the stiffness r which could be very large

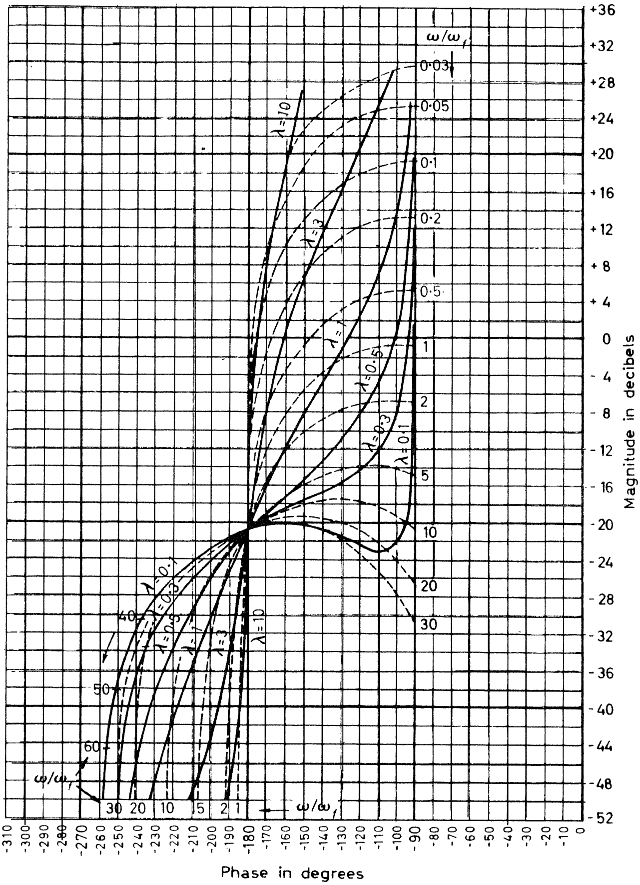


Figure 7.31. Curves of

$$\frac{1}{\frac{j\,\omega}{\omega_f}\left(1+\frac{r}{r_h}+\lambda^2\frac{j\,\omega}{\omega_f}+\frac{r}{r_h}\lambda^2\frac{j^2\,\omega^2}{\omega_f^2}\right)}$$

for $r/r_h = 0{\cdot}1$ and different values of λ

locally and also to cover the case where the fluid is very compressible so that the hydraulic stiffness, r_h, is very small, the frequency at which the phase difference reaches 180° will no longer be

$$\omega_c = \sqrt{\frac{2\,BS^2}{V_t\,m}} = \sqrt{\frac{r_h}{m}}$$

but

$$\omega_c' = \sqrt{\frac{r + r_h}{m}}$$

The corresponding amplitude of the transfer function is

$$A_{\omega'_c} = \frac{\omega_f}{\dfrac{m}{r}\,\omega_f\,\omega_c'^2} = \frac{r}{r + r_h}$$

For positive values of the stiffness r (r_h always being positive), this amplitude is always less than unity and is *independent of the open-loop gain*, ω_f.

We may draw the important conclusion that *a force servocontrol with pressure detection is theoretically always stable for all values of the open-loop gain.*

(*b*) *Stability margin*—We know that in practice the absolute condition of stability is not always sufficient, and a certain stability margin should be allowed.

Gain margin—To avoid amplitude rise in the closed loop, we must allow a gain margin of 6 dB at $\omega = \omega'_c$, so that

$$A_{\omega'_c} \leqslant \tfrac{1}{2}$$

i.e.

$$\boxed{r \leqslant r_h}$$

Phase margin—The definition of phase margin leads to complicated calculations and results. We can, however, get a simple result which is very satisfactory in practice by taking advantage of the fact that, for a certain value of the gain, all the transfer loci pass through a single point.

Suppose that the open-loop gain, ω_f, is equal to $\sqrt{r/m}$, i.e. has the same value as the mechanical resonant angular frequency of the assembly consisting of the load + ram without oil; eqn. (76) then becomes

$$H_1 = \frac{1}{\dfrac{p}{\omega_f}\left(1 + \dfrac{r}{r_h} + \dfrac{p}{\omega_f} + \dfrac{r}{r_h}\,\dfrac{p^2}{\omega_f^2}\right)} \tag{77}$$

Phase difference: $\phi = 90° + \tan^{-1}\dfrac{1}{1 + r/r_h - r/r_h} = 135°$

Amplitude: $A = 1/\sqrt{2} = -3$ dB

Thus when $\omega_f = \sqrt{r/m}$, for all values of r_h, the transfer locus passes through the point

$$\begin{cases} \phi = -135° \\ A = -3\,\text{dB} \\ \omega = \omega_f \end{cases}$$

If the transfer function is a regular third-order function, the phase margin will be greater than 45° for $\omega_f \geqslant \sqrt{r/m}$ (see *Figures 7.31* and *7.32*). This gives the following fundamental result:

A force servocontrol with pressure detection is stable in practice if its open-loop gain is less than, or equal to, the resonant angular frequency of the load, $\sqrt{r/m}$, provided that the hydraulic stiffness, r_h, is greater than the mechanical stiffness, r, of the load.*

This result requires some comment.

(*1*) *Insensitivity of the servocontrol to the bulk modulus, B*—Contrary to the result obtained for the position servocontrol, the force servocontrol is generally

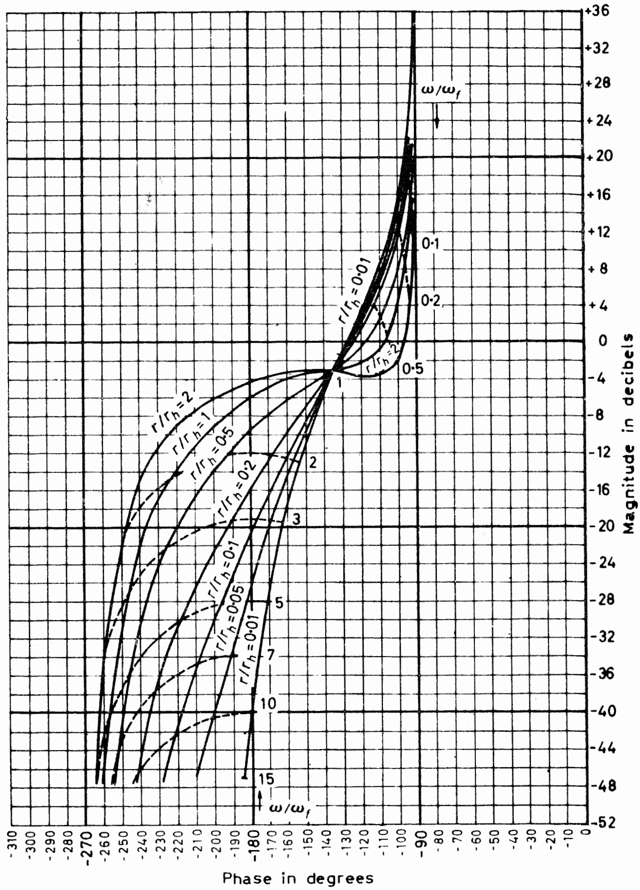


Figure 7.32. Curves of

$$\frac{1}{\dfrac{j\omega}{\omega_f}\left(1+\dfrac{r}{r_h}+\dfrac{j\omega}{\omega_f}+\dfrac{r}{r_h}\dfrac{j^2\omega^2}{\omega_f^2}\right)}$$

for different values of r/r_h

* If the mass of the movable part of the ram cannot be neglected in comparison with the mass of the load, it should be included in m.

insensitive to variations of the bulk modulus, B. Since r_h is generally very much greater than r in a hydraulic servocontrol, the gain margin condition for stability is normally satisfied. This leaves only the phase margin condition which is not affected by B.

(2) *The use of compensating networks*—We may well ask whether a compensating network could not give $\omega_f > \sqrt{r/m}$ without affecting the stability.

Put $\omega_f = \lambda\sqrt{r/m}$, with $\lambda > 1$. If $r_h/r \gg 1$, the transfer function, H_1, becomes

$$H_1 \simeq \frac{\omega_f}{p\left(1 + \dfrac{m}{r}\,\omega_f p + \dfrac{m p^2}{r_h}\right)} = \frac{\omega_f}{p\left(1 + 2\zeta\dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right)}$$

where

$$\omega_c = \sqrt{\frac{r_h}{m}}, \quad \text{and} \quad \zeta = \frac{1}{2}\,\frac{m}{r}\,\omega_f\,\omega_c = \frac{1}{2}\,\lambda\sqrt{\frac{r_h}{r}};$$

ζ will thus be much greater than 1.

Under these conditions, H_1 is, up to high frequencies, approximately equal to

$$\frac{\omega_f}{p\left(1 + \dfrac{m}{r}\,\omega_f p\right)} = \frac{1}{\dfrac{p}{\omega_f}\left(1 + \lambda^2\dfrac{p}{\omega_f}\right)}$$

A phase-advancing network of transfer function

$$\frac{1 + \lambda^2 p/\omega_f}{1 + p/\omega_f}$$

enables us to re-establish the necessary phase margin while conserving the gain $\lambda\sqrt{r/m}$. This gives the following result, which has no equivalent in the position servocontrol:

The addition of a phase-advancing network to a force servocontrol enables the open-loop gain to be increased beyond $\sqrt{r/m}$ while retaining the same stability margin. The higher the value of r_h/r, the higher the increase that can be made.

N.B. Since the regulation parameter K, the valve gain, appears not only in the numerator of the open-loop transfer function but also in a term of the denominator, the usual procedures for adjusting the gain, for example by translating Nichols curves, cannot be used.

(3) *Pneumatic force servocontrols*—Owing to the high compressibility of air, pneumatic position servocontrols were said to have considerably poorer performances than hydraulic ones. This is not so for force servocontrols. Suppose a pneumatic servocontrol has a maximum force F_M, maximum half travel z_M,

volumetric coefficient k_v and operates at a point where the stiffness of the load is r. We have

$$\frac{r}{r_h} = \frac{V_t r}{2\,BS^2} = \frac{P_1}{2\,B}\,\frac{k_v r}{F_M/z_M}$$

If it operates adiabatically

$$PV\gamma = C$$

where C is a constant and

$$\frac{\Delta P}{P} = -\gamma\,\frac{\Delta V}{V}$$

wherefrom

$$B = \frac{-\,\Delta P}{\Delta V/V} = \gamma P$$

With $P = P_1/2$, we have

$$\frac{r}{r_h} = \frac{1}{\gamma}\,k_v\,\frac{r}{F_M/z_M}$$

With $\gamma = 1{\cdot}4$ and if, for example, $k_v = 1{\cdot}4$, the gain margin will be 6 dB as before, provided that $r/r_h \gg 1$, i.e. $r \leqslant F_M/z_M$. This condition is independent of the gain and the supply pressure, but nevertheless it is difficult to fulfil without considerably oversizing the ram, and we have to add the condition $\omega_f \leqslant \sqrt{r/m}$.

A normal phase-advancing network gives no improvement in the case of this pneumatic servocontrol since, although it would improve the phase margin, it would not improve the gain margin, which is more critical here, except in the case of a rotational pneumatic motor for which k_v can be very much less than 1 (cf. Section 7.9.3).

(*4*) *Effect of variations of r*—The condition for stability in the absence of a compensating network has been shown to be $\omega_f \leqslant \sqrt{r/m}$. With

$$\omega_f = \frac{K\,r}{k\,(\mu - 1)\,S}$$

the valve gain, K, must therefore satisfy the equation

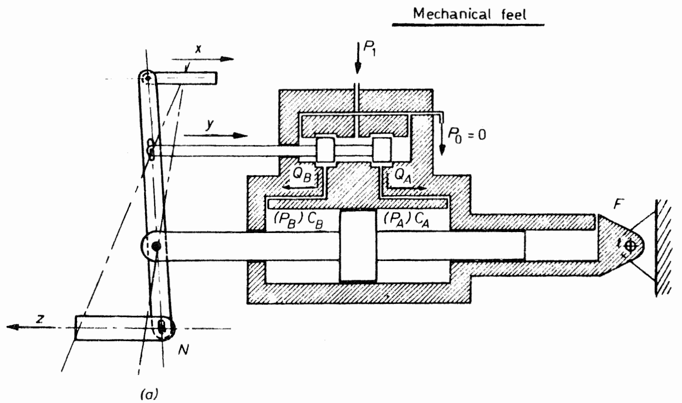
$$K \leqslant \frac{k\,(\mu - 1)\,S}{\sqrt{m\,r}}$$

The gain decreases as r increases. *The estimation of K should be made using the maximum possible value of r*. Note that, for a given value of K, the response speed increases with r, since to obey a given order (i.e. to go from one given opposing force to another), the travel, and therefore the corresponding flow, decreases as r increases.

7.9.5. SERVOCONTROLS WITH FEEL AND ARTIFICIAL FEEL

We must not confuse the force servocontrol without position feedback with the servocontrol having 'feel'. The servocontrol with feel is a position servocontrol in which, by some method, a force proportional to the resistance of the load is applied to the control linkage. In an aircraft, for example, the purpose of this is to enable the pilot to 'feel' the control and to allow him to use the reflexes which he developed when using similar aircraft not fitted with servocontrols.

The analysis follows the same general lines as that of position servocontrols. *Figure 7.33* shows two different designs.



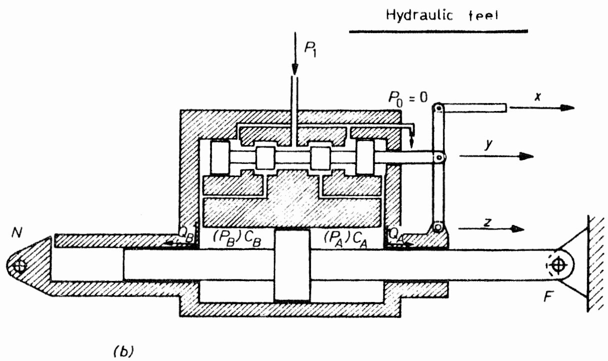


Figure 7.33

Many manufacturers consider the reversibility of the servocontrol with feel to be a disadvantage and prefer to use an artificial feel, in which the force applied to the control linkage is completely fabricated, without any reference to the actual load. This artificial force, F_a, can be a linear function of the control position (*Figure 7.34a*), linear with a force dead zone (*Figure 7.34b*) or it can be a function of another parameter, e.g. dynamic pressure, P_D, in addition to being a function of the control position (*Figure 7.34c*).

7.10. CONCLUSION

At an equal, or even higher, speed the hydraulic servocontrol develops a power considerably greater than that of the electric servocontrol. This is mainly due to the low mass of the moving parts of the ram or of the rotary actuator*. Hydraulic servocontrols are therefore used when extreme requirements of power and speed are imposed at the same time.

When the inertia of the load is high, the hydraulic servocontrol may become unstable. In addition, the risk of instability can be aggravated considerably by the presence of air in the circuit or by insufficient rigidity of the attachment and connecting fixtures. There are, however, a number of methods available for stabilization.

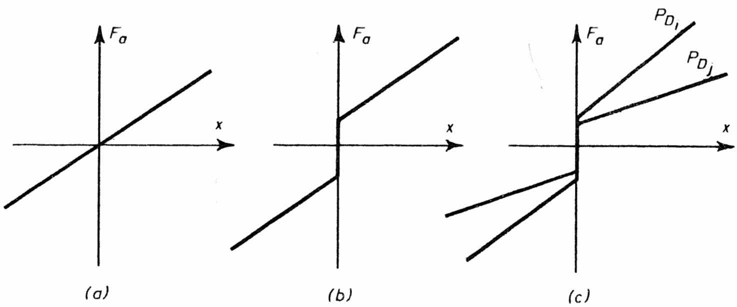


Figure 7.34

For installations at a fixed station, and in general for equipment where economy of energy is not of prime importance, internal leakage methods are used because of their simplicity and effectiveness. The leakage flow is often located in the control valve, even though it uses more energy than leakage in the ram, since it can be used with less accurate, and therefore much less expensive, control valves.

If economy of energy is important, stabilization by control valve dead zone is used, provided it is compatible with the static accuracy required. This is often the case for manned vehicles.

When the specifications for energy and static accuracy are stringent, secondary feedback methods must be considered. Their introduction is made easier when servo valves are used, since the feedback signal can be electrical. Some manufacturers are even producing servo valves with internal secondary hydraulic feedback.

Finally, we should note that the design and practical construction of hydraulic systems is becoming less and less difficult as methods of hydraulic analysis are developed and specialized basic hydraulic components become more common on the market.

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* In fact, hydraulic motors can be operated so that the material is stressed almost to its elastic limit,whereas the physical limitation of the inductance of electric motors results in very low maximum stresses.

APPENDIX 7.1

STABILIZATION BY DYNAMIC PRESSURE FEEDBACK

(1) INTRODUCTION OF A SECONDARY FEEDBACK

Consider, for example, an electrohydraulic position control of input e (V), output z (cm) and consisting of

an amplifier: K_1

a servo valve: $$\frac{K_2}{1 + \alpha p + \beta p^2 + \cdots} = \frac{K_2}{A(p)} \quad \text{(see Section 10.9)}$$

a ram (effective area, S): $$\frac{1}{Sp\left[1 + 2\zeta\dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right]} = \frac{K_3}{p\,B(p)} \quad \text{(cf. Section 7.5)}$$

and a position detector: K_4.

Putting $K_1K_2K_3K_4 = K$, we can draw the block diagram (*Figure 7.35*).

The addition of a secondary feedback, $kg(p)$, changes the diagram to that shown in *Figure 7.36a*, which can be reduced to the form shown in *Figure 7.36b*.

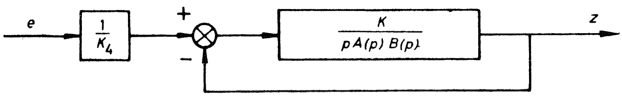


Figure 7.35

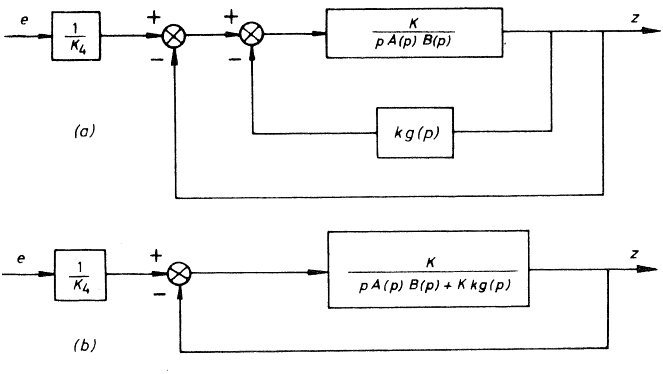


Figure 7.36

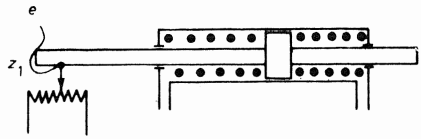
(2) PRESSURE FEEDBACK WITHOUT FILTER

Suppose that the pressures P_A and P_B are applied to the two sides of a detector piston having an effective area s_1, a position z_1, restoring springs of total stiffness r_1, and controlling a linear potentiometer of gain k_1 where the slider potential is e' (*Figure 7.37*).

Combining $(P_A - P_B)s_1 = r_1 z_1$ and $e' = k_1 z_1$ with $(P_A - P_B)S = (r + mp^2)z$ gives

$$kg(p) = \frac{1}{K_4}\frac{e'}{z} = \frac{1}{K_4}\frac{k_1}{r_1}\frac{s_1}{S}(r + mp^2)$$

Figure 7.37



If $A(p)$ is approximately equal to unity up to $\omega = \omega_c$, the open loop of *Figure 7.36b* can be expressed by

$$\frac{K}{\varepsilon' + p\,[1 + 2(\zeta + \zeta')\,p/\omega_c + p^2/\omega_c^2]}$$

where

$$\zeta' = \frac{\omega_c}{2}\frac{K}{K_4}k_1\frac{s_1}{S}\frac{m}{r_1}$$

(supplementary reduced damping coefficient) and

$$\varepsilon' = \frac{K}{K_4}k_1\frac{s_1}{S}\frac{r}{r_1} = \zeta'\frac{2}{\omega_c}\frac{r}{m}$$

(static error term).

If, in addition, ζ and ϵ' are negligible, we can obtain directly a simple condition for stability. If

e_M = position detector voltage for $z = z_M$ $(e_4 = K_4 z_M)$

e'_M = pressure detector voltage for $\Delta P = P_1$ $(e'_M = k_1 s_1 P_1/r_1)$

For $\omega = \omega_c$, $\phi = 180°$ and $A = -k/2p\omega_c K/2\zeta'\omega_c$

The gain margin will be greater than 6 dB if $K/2\zeta'\omega_c < \frac{1}{2}$, i.e. if

$$\boxed{\frac{e'_M}{e_M} > \frac{P_1}{B}k_v}$$

(3) PRESSURE FEEDBACK WITH HIGH-PASS FILTER

To avoid the static error, we could use a high-pass filter which enables the secondary pressure feedback to be conserved except at high frequencies (in the neighbourhood of ω_c). A simple arrangement is shown in *Figure 7.38*.

The function of the high-pass filter is physically obvious: in stable operation, there is no flow through the orifice O. The pressure on both sides of the piston is P_A, so that there is no detection even if $P_A \neq P_B$. In rapid transient conditions,

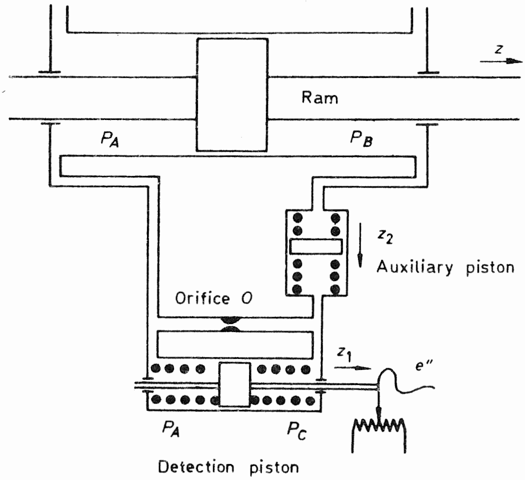


Figure 7.38

the orifice O produces a loss of head and the auxiliary piston has less effect. The pressure P_C tends to follow variations in P_B, so that there is detection.

Forming the equation—In addition to the symbols used previously, we have

Auxiliary piston: area s_2, stiffness r_2, position z_2
Orifice O: flow coefficient a ($a = Q/\Delta P$)
New secondary feedback: e''

The basic equations are:

Equilibrium of the detection piston:

$$(P_A - P_C)\, s_1 = r_1 z_1 \tag{1}$$

Equilibrium of the auxiliary piston:

$$(P_B - P_C)\, s_2 = r_2 z_2 \tag{2}$$

Flow equation:

$$a\,(P_A - P_C) + s_1 \frac{dz_1}{dt} + s_2 \frac{dz_2}{dt} = 0 \tag{3}$$

By eliminating $(P_A - P_C)$, $(P_B - P_C)$ and z_2, we obtain the relationship between z_1 and $(P_A - P_B)$

$$(P_A - P_B) = z_1 \frac{r_1}{s_1}\left[1 + \frac{r_2}{r_1}\frac{s_1^2}{s_2^2} + a\frac{r_2}{s_2^2 p}\right]$$

Since $z_1 = e''/k_1$ and $(P_A - P_B) = (1/S)(r + mp^2)z$, this becomes

$$k\,g(p) = \frac{1}{K_4}\frac{e''}{z} = \frac{1}{K_4}\frac{k_1}{r_1}\frac{s_1}{S}(r + m\,p^2)\frac{1}{\left(1 + \dfrac{r_2}{r_1}\dfrac{s_1^2}{s_2^2}\right)}\left[\frac{p}{p + \dfrac{a}{\dfrac{s_2^2}{r_2} + \dfrac{s_1^2}{r_1}}}\right]$$

Introducing e'/z, which was calculated in the previous Section, we have finally

$$\boxed{\frac{e''}{z} = \frac{e'}{z} \cdot \frac{1}{\left(1 + \dfrac{r_2}{r_1} \dfrac{s_1^2}{s_2^2}\right)} \cdot \left[\frac{p}{p + \dfrac{a}{\dfrac{s_2^2}{r_2} + \dfrac{s_1^2}{r_1}}}\right]}$$

This expression for e''/z could obviously be inserted immediately in the reduced open loop of *Figure 7.36b*. A direct examination of e''/z, however, is much more useful and more in keeping with modern methods of analysing servo systems. It can be seen that e''/z is the product of three terms:

1) $$\frac{e'}{z}$$

the secondary pressure feedback without filter, whose effect has been dealt with above;

2) $$\frac{1}{1 + \dfrac{r_2}{r_1} \dfrac{s_1^2}{s_2^2}}$$

a numerical constant which, for technological reasons, will always be in the region of $\frac{1}{2}$;
(*c*)

3) $$\frac{p}{p + \dfrac{a}{\dfrac{s_2^2}{r_2} + \dfrac{s_1^2}{r_1}}}$$

which can be written as $p/(p+\omega_0)$ where

$$\omega_0 = \frac{a}{\dfrac{s_2^2}{r_2} + \dfrac{s_1^2}{r_1}}$$

and which is approximately equal to

$$\begin{cases} 0 & \text{for} \quad \omega \ll \omega_0 \\ 1 & \text{for} \quad \omega \gg \omega_0 \end{cases}$$

The secondary feedback is therefore suppressed at low frequencies but conserved at high frequencies, with a multiplying constant of about $\frac{1}{2}$. If we therefore make $\omega_0 < \omega_c$ (by increasing k_1), the action of the feedback without filter is retained, since it is effective in the region of $\omega = \omega_c$; the static error is eliminated, being replaced at the most by a certain 'slowness at the end' in motion towards a final equilibrium position.

In the design of this equipment, the values of r_1, r_2, s_1 and s_2 are mainly determined by technological considerations; for given values of these parameters, the size of the orifice (i.e. the value of a) determines ω_0 and therefore enables us to make a compromise between the degree of stabilization and the rapidity of completing the transient motion.

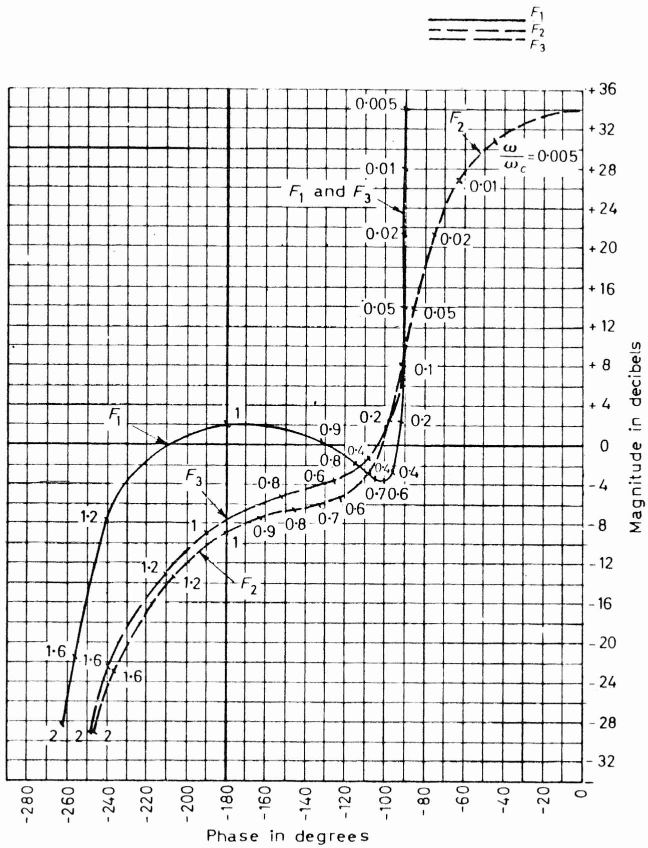


Figure 7.39

Figure 7.39 shows

(a) the open-loop transfer function

$$F_1 = \frac{0{\cdot}25\,\omega_c}{p\left[1 + 2\,\zeta\,\dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right]} \qquad \text{(with } \zeta = 0{\cdot}1\text{)}$$

of an unstable hydraulic position servocontrol (servo valve transfer function approximately unity).

(b) the open-loop transfer function of the same servocontrol, stabilized by a simple pressure feedback

$$F_2 = \frac{0{\cdot}25\,\omega_c}{p\left[1 + 2\,\zeta\,\dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right] + \left[\varepsilon' + 2\,\zeta'\,\dfrac{p^2}{\omega_c}\right]}$$

with $\zeta' = 0{\cdot}25$ and

$$\varepsilon' = \frac{2\,\zeta'}{\omega_c}\,\frac{r}{m} = 2\,\zeta'\,\omega_c\,\frac{P_1}{2\,B}\,\frac{k_v}{k_s} = 0{\cdot}5 \times 10^{-2}\,\omega_c$$

($k_s = k_v$; $B = 10{,}000$ kg/cm^2; $P_1 = 200$ kg/cm^2).

(*c*) the open-loop transfer function of the same servo, stabilized by a pressure feedback with high-pass filter

$$F_3 = \frac{0{\cdot}25\,\omega_c}{p\left[1 + 2\,\zeta\,\dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right] + \left[\varepsilon' + 2\,\zeta'\,\dfrac{p^2}{\omega_c}\right]\left[\dfrac{p}{p + \omega_c/4}\right]}\,.$$

APPENDIX 7.2

PHOTOGRAPHIC RECORDINGS OF RESPONSE TESTS

On lengths of photographic paper which are unrolled at constant speed, are recorded simultaneously

the input, e [electrical on (1) and (2), position on (3) and (4)];

the output, s [position of the servocontrol].

Note that, for the ordinates, neither the scales nor the origins coincide, being determined by a previous calibration.

Figure 7.40, (1) and (2). Electro-hydraulic actuators of missile control surfaces with potentiometric position detection

Trace (1): frequency, 20 c/s; amplitude: $e = e_{max}/10$; the traces have been used to determine the reduction in amplitude and the phase difference.

Trace (2): frequency, 50 c/s; amplitude: $e = e_{max}/20$. The irregularities in the output curve are repeated exactly in each cycle. They arise from the potentiometer of the servocontrol, which was simultaneously used to measure the output.

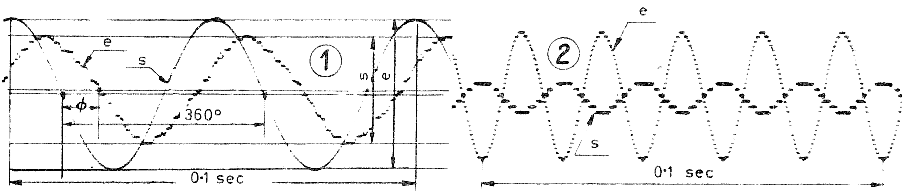
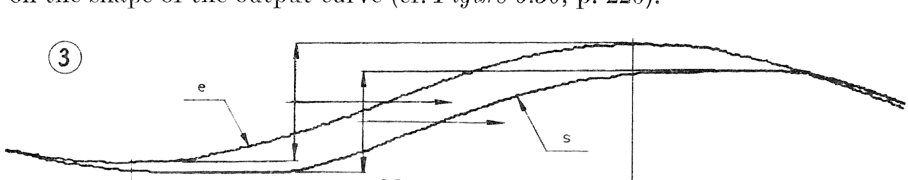
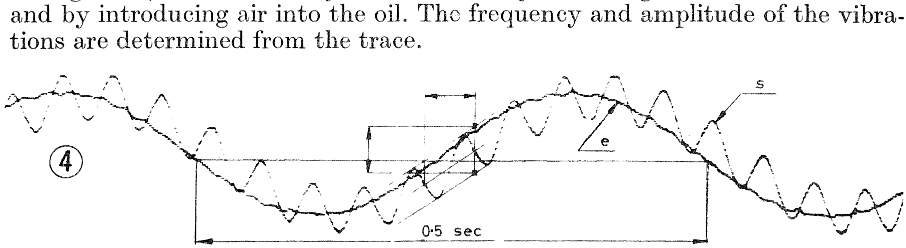


Figure 7.40

Trace (3): *Aircraft servocontrol stabilized by a dead zone in the control valve.* Frequency: 1 c/s; amplitude: $e = e_{max}/30$. Note the effect of the valve dead zone on the shape of the output curve (cf. *Figure 6.30*, p. 220).



Trace (4): *Unstable aircraft servocontrol* (frequency and amplitude as preceding trace). The instability was obtained by increasing the inertia of the load and by introducing air into the oil. The frequency and amplitude of the vibrations are determined from the trace.



APPENDIX 7.3

MODERN METHODS OF STABILIZING SERVOCONTROLS

The general problem of stabilization of servo systems has been dealt with in Section 7.7. The stabilizing effect of structural feedback was considered in Sections 7.8.1 and 7.9.1. Finally, an example of stabilization using a high-pass filter, and therefore having no static error, was given in Appendix 7.1.

These pages, published in the first French edition, were written at a time when stabilization problems were only beginning to be assessed. Since then, methods of stabilization have been developed considerably and, although there is nothing to add to or withdraw from the general theory given in the Sections quoted, it may be useful to indicate methods currently being adopted by the manufacturers.

HIGH-PASS FILTER (DYNAMIC) METHODS

All known methods introduce a static error, except those based on the detection of acceleration which in any case are not used because of their technological complexity.

It has been shown (Section 7.7) that this static error is proportional to the damping effect obtained and, for constant damping, independent of the retained principle (control valve leakage, ram leakage, secondary pressure feedback).

The static error can be extremely detrimental to the system, particularly in the servocontrols of transonic aircraft. For this reason all modern stabilization methods employ a high-pass filter: its use enables them to be called 'dynamic'.

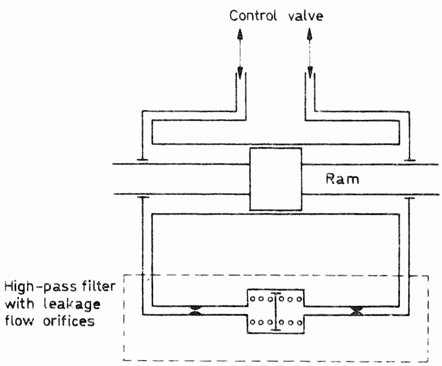


Figure 7.41. Stabilization by dynamic leakage flow between the chambers of the ram: high-pass filter in series

Certain methods of stabilization are obviously well suited to the introduction of a high-pass filter. Consider, for example, the arrangement shown in *Figure 7.41* where a filter is mounted in series in the leakage flow between the two chambers of a hydraulic ram. Its mode of operation is very simple, since a piston with an elastic restoring force provided by springs (see *Table 5.2*) behaves in a manner similar to an electrical condenser, i.e. it allows high-frequency alternating flows to pass but blocks direct flows and reduces low-frequency alternating flows. This device, however, is very seldom used in practice, since the piston and spring component, which has itself to accommodate the volume of oil whose movement is necessary for stabilization, becomes very bulky, its size increasing as the required operating pressure is increased.

In the more acceptable designs, the high-pass filter has the function of acting as a *calculating element*, and its size can, to a certain extent, be chosen arbitrarily.

In a practical lay-out there is always a fundamental element consisting of a movable spool, M, whose position y_1 is related to the pressure difference, ΔP, between the chambers of the ram by an equation of the form

$$\frac{y_1}{\Delta P} = \frac{p}{p+\omega_0}$$

This equation, characteristic of the high-pass filter, is obtained by a combination of the filter with elastic restoring force and one or more calibrated orifices (cf. the RC circuit of electronics).

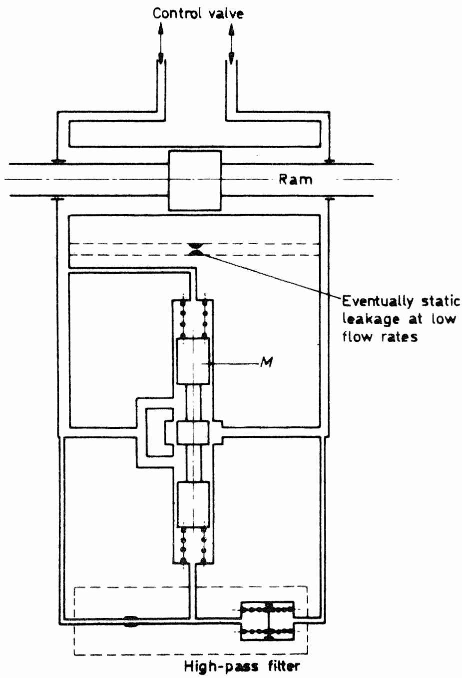


Figure 7.42. Stabilization by dynamic leakage flow between the chambers of the ram: auxiliary control valve, high-pass filter used as calculator

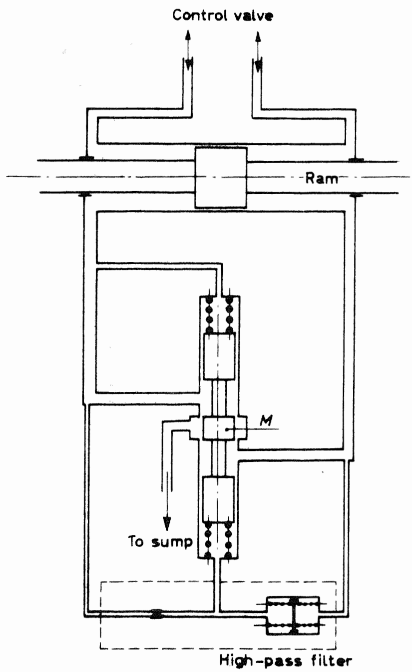


Figure 7.43. Stabilization by dynamic leakage flow to sump

Figures 7.42 and *7.43* show examples of some of the first designs. Under steady conditions the spool M is in the middle position, because its end faces are subjected to equal pressures, in this case the pressure in the left-hand chamber of the ram. Under transient conditions, the pressure applied to the upper end of the spool M remains the same as that of the left chamber of the ram, but the pressure on its lower end tends to that in the right-hand chamber as the frequency is increased.

Another actual design is shown in *Figure 7.44*. This is more complex, having a staged spool, but it has the advantage, although somewhat theoretical, of not favouring the pressure in one or other of the chambers of the ram, as was the case in the previous design. Nevertheless it still incorporates a piston with elastic restoring force and an orifice (or orifices).

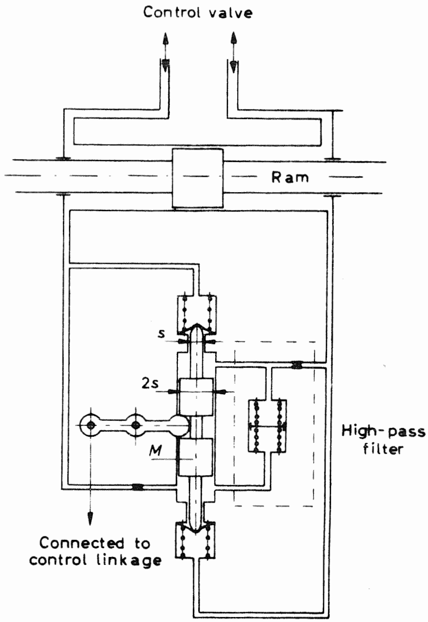


Figure 7.44. Stabilization by dynamic pressure feedback transformed to a position signal at the control valve

In an attempt to increase reliability by decreasing the number of moving parts, systems have been proposed in which the piston with restoring force has been replaced by two non-active storage volumes (see *Table 5.2*). But, unless dangerously small calibrated orifices are used, these storage volumes again result in a heavy and bulky system.

Thus modern methods of stabilization differ more by the principle involved, i.e. by the use made of the movement of the spool M, than by the design of the high-pass filter itself.

DYNAMIC LEAKAGE FLOW BETWEEN THE CHAMBERS OF THE RAM

Stabilization by static leakage flow between the chambers was dealt with in Section 7.7.1.1.

Since the simple dynamic leakage method illustrated in *Figure 7.41* is so bulky, some modern aircraft now being developed employ the layout shown in *Figure 7.42* where the dynamic leakage between the chambers of the ram is controlled by a movable spool or slider.

This device, however, has some disadvantages. The leakage flow is proportional to the product of the cross-sectional area and the square root of the pressure difference ΔP between the two chambers and, since the cross-sectional area is (at high frequency) itself proportional to ΔP, the leakage flow is finally proportional to $(\Delta P)^{3/2}$. The system is no longer linear and its analysis, e.g. by the methods given in Section 6.10, indicates that an oscillation limit of low amplitude may exist. This oscillation can be eliminated by the addition of a very small static leakage flow, at the expense of introducing a very small static error (see *Figure 7.42*).

DYNAMIC LEAKAGE TO THE SUMP

In another design, the leakage flow between the chambers of the ram is replaced by a leakage from each chamber to the sump. At first sight it is apparent that this system (*Figure 7.43*) is not linear either. It can, however, be shown to be equivalent to a linear system in which the spool M would be a secondary four-way control valve, with a gain equal to half that of a three-way control valve, which is what it actually is.

This fact enables us to analyse the stability of these devices simply but completely and also to programme them easily on an analogue computer.

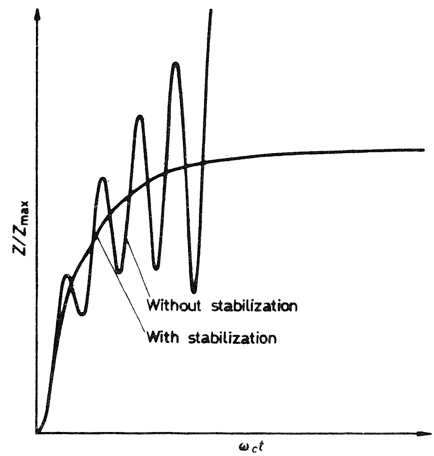


Figure 7.45. Analogue computer curves

Figure 7.45 shows the response of a modern servocontrol to a large amplitude step, obtained on an analogue computer, both without any stabilizing device and with the dynamic method of leakage flow to the sump. The stabilization device is obviously very effective. The only disadvantage is the fairly sensitive nature of the adjustment of the spool M, as an error here entails a permanent consumption of oil.

PRESSURE DETECTION TRANSFORMED TO SECONDARY POSITION SIGNAL AT THE CONTROL VALVE (*Figure 7.44*)

This is the layout of the stabilization method shown by *Figure 7.16* to which a high-pass filter has been added.

Its advantage is that it does not introduce a supplementary oil consumption, while its disadvantage is the resulting technological complexity. It has a mechanical output which is a position signal and which is passed to the main control valve by means of a pivoted lever and connecting linkage. In order that the system be completely independent of the frictional forces, the stabilizing device has to have a certain mechanical impedance, sufficiently large for the force available at the output to be large compared with the friction.

It is beneficial to immerse the lever in oil; this considerably reduces the friction present after the introduction of the secondary signal, i.e. that which is likely to affect the performance, but this, of course, introduces further technological difficulties.

Equations for determining the required size of components

(*a*) Determination of the *level of leakage flow* necessary for the introduction of damping, ζ, defined by

$$\frac{Spz}{Q} = \frac{1}{1+2\zeta(p/\omega_c)+p^2/\omega_c^2}$$

ζ is related to the flow coefficient, A, by the equations

$$2\zeta = \frac{A}{S^2}m\omega_c \quad \text{or} \quad 2\zeta = \frac{A}{S}\sqrt{\frac{2Bm}{V_t}}$$

where the flow coefficient A is given by

$$A = \frac{q}{P_A - P_B} \quad \text{(for } \textit{Figures 7.41} \text{ and } \textit{7.42}\text{)}$$

$$A = \frac{1}{2}\frac{q}{|P_A - P_B|} \quad \text{(for } \textit{Figure 7.43}\text{)}$$

(*b*) Determination of the *high-pass filter*

$$\frac{Spz}{Q} = \frac{1}{1+[p/(p+\omega_0)]2\zeta(p/\omega_c)+p^2/\omega_c^2}$$

$$\omega_0 = \frac{ra}{s^2}$$

where

a = flow coefficient of the orifice
r = stiffness of piston return springs
s = effective area of piston

If the device is to be effective, ω_0 must be significantly less than ω_c.

8

ELECTROHYDRAULIC SERVO SYSTEMS

8.1. INTRODUCTION

The electrohydraulic servo system has been introduced only recently, but its scope is expanding rapidly and will certainly expand much further in the course of the next few years.

We must first find out why this system was developed and what requirement it fulfilled.

In the previous Chapter, an analysis was made of a hydraulic servo system with a mechanical input. Suppose that in a certain application (automatic machinery, guided missiles, etc.), the input is electrical in form (voltage or current) rather than mechanical (position or force). The electrical input could obviously be transformed into a mechanical position by means of an electric actuator (*Figure 8.1a*) or servomechanism (*Figure 8.1b*).

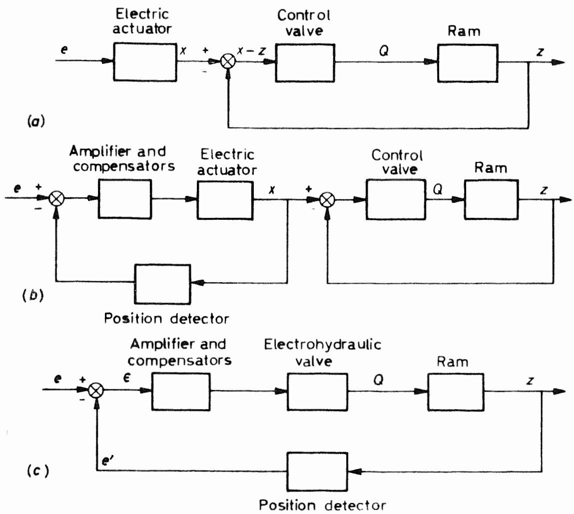


Figure 8.1

This solution is used, for example, in the automatic pilots of aircraft which are presently in service. In the 'Caravelle', for example, the automatic pilot controls an electric actuator which, in parallel with the human pilot, moves the spool of a hydraulic servocontrol.

But the electrical input, e, could also be compared with an electrical form, e', of the output (for example, the position z of the ram) and the error, $\epsilon = e - e'$, could be amplified, compensated and transformed into hydraulic flow (*Figure 8.1c*).

This arrangement is found in all rockets with hydraulic controls and is being used in the Concorde supersonic air liner. It has a number of advantages:

(*a*) it is *very flexible*; the gain of the hydraulic control loop may be adjusted very easily, and compensation may take the form of electrical circuits rather than the more complex hydraulic ones. The main input signal may be modified or have secondary signals, e.g. stabilizing or safety functions, imposed on it. This is why this type of chain is used for the control of modern supersonic aircraft in which it is necessary to add gyrometric stabilization signals to the normal pilot's signals and to ensure that a new alternative connection is made automatically in the case of the failure of an element of the chain (fail-safe system).

(*b*) it has a very *short response time*. When it is necessary to combine rapid response with high power levels, the hydraulic system is the best, but when the energy level is low, the electrical system can have an extremely high speed of response. Thus, by using an electrohydraulic system, in which the electrical components handle only the low energy signals, the speed of response for the whole assembly will be a maximum. This is why this type of chain is used for electrohydraulic rocket-steering systems.

Electrohydraulic servo systems combine the advantages of electrical systems, such as flexibility, ease of adjustment, the facility to carry out logic functions such as $+$, $-$, or, etc., and mathematical functions such as multiplication, division, integration, differentiation, etc., as used in computers, with those of hydraulic systems, resulting in high-performance systems with quick response, high power-to-weight ratio and high reliability.

In view of these qualities, we may well ask why electrohydraulic servo systems were not developed earlier and used more widely. There are several reasons. The main one was the difficulty of manufacturing the essential component of the chain, a valve which transforms an electrical signal into a hydraulic flow. This component is called a servo valve. The first real servo valves were designed only a few years ago to meet rocket-steering requirements. Another reason was the new and individual nature of the electronic problems encountered in closing the loop—problems which called for highly qualified specialists to examine each specific case. Finally, unless special precautionary measures regarding increase of weight are taken, the electrohydraulic chain is less dependable than a purely hydraulic one—a disadvantage which is likely to persist for some time.

A wide variety of different types of servo valves is now available commercially at prices which are certainly still very high but which will drop considerably when mass-production methods being investigated by several manufacturers are brought into use.

Soon, then, the main obstacle to the wider use of electrohydraulic servo systems will be the fact that engineers are not aware of the scope and variety of servo valves available or of the electronic techniques for closing the loop.

The following Sections will therefore deal with these two points.

The actual *performance* of hydraulic servo systems is sufficiently similar to that of purely hydraulic systems to be covered by a single example, given in Chapter 9.

8.2. THE ELECTROHYDRAULIC SERVO VALVE

8.2.1. DEFINITION

The electrohydraulic servo valve is a component which, in response to an *electric current input*, provides a *proportional output* in the form of a *hydraulic flow* (or, sometimes, *pressure*).

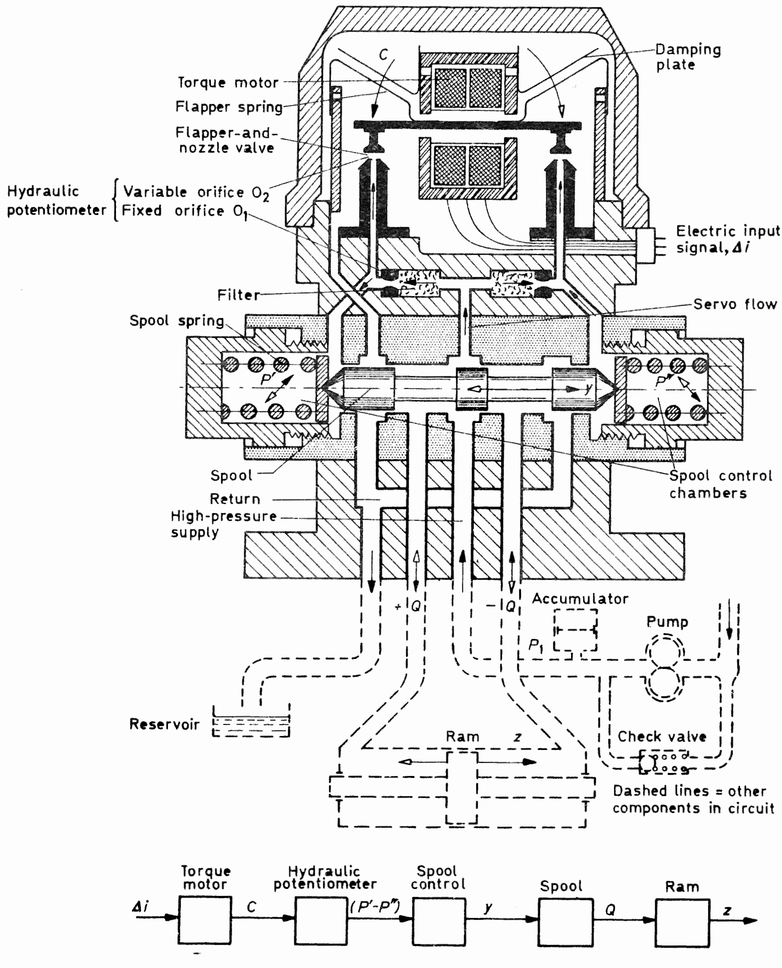
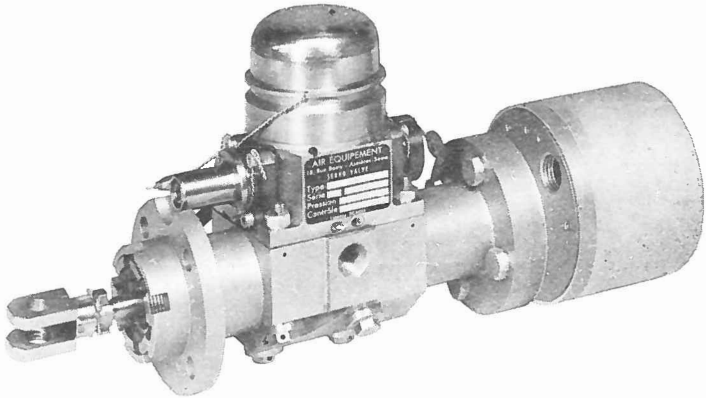
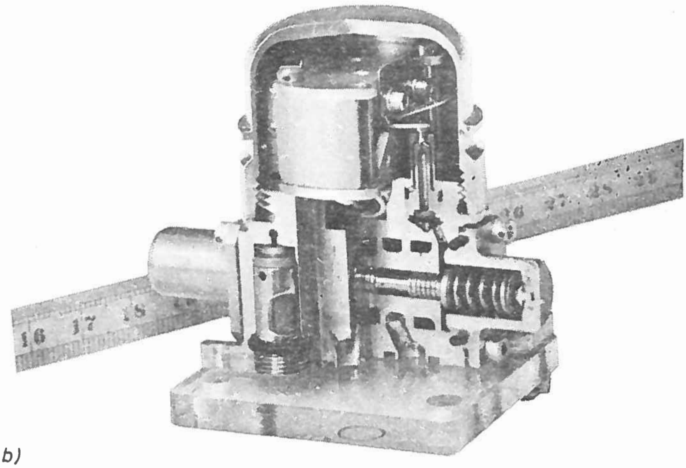


Figure 8.2

There is widespread confusion, even in manufacturers' catalogues, as to the difference between servo valves and electrovalves. In principle, the term electrovalve is reserved for *on–off* components whose input is electric current and whose output is hydraulic flow.



(a)



(b)

Plate 1. (*a*) Electrohydraulic ram fitted with Air Equipement servo valve 30261; (*b*) section through servo valve

The types of servo valve now available commercially represent a wide variety of designs based on different principles.

A servo valve consists basically of a hydraulic valve which is controlled by an electric actuator, although many are more complex, having several hydraulic stages and various devices for internal feedback or stabilization.

We shall first consider the mode of operation of a fairly simple modern servo valve, then review the main types available and finally consider the more important points in the selection of a particular design.

8.2.2. DESCRIPTION

The servo valve chosen as an example is the Air Equipement type 30261, manufactured under licence to Bendix. It is a symmetrical four-way flow servo valve with two hydraulic stages (*Figure 8.2, Plate 1*).

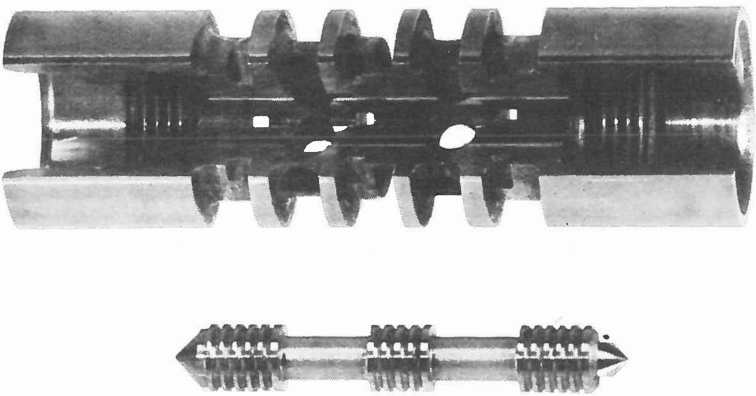


Plate 2. Control spool of Air Equipement servo valve 30261

8.2.2.1. *Principles of operation*

The input current, Δi, entering the servo valve is applied to a push-pull torque motor having a permanent magnet and two sets of windings, as shown at the top of the sectional elevation in *Figure 8.2*.

The current through the windings, which is i_0 through each under steady conditions, becomes $i_0+\Delta i/2$ and $i_0-(\Delta i/2)$, thus applying a torque, C, to the floating flapper plate, P, which fits across the torque motor:

$$[\Delta i \rightarrow C \ldots]$$

When there is no input current, this plate is in equilibrium under the action of its supporting springs and the pressure forces of the fluid jets issuing from the variable orifices O_2: it therefore maintains equal pressures P_0' and P_0'' in the two spool control chambers.

Under the action of the torque C the flapper plate tends to close one of these variable orifices O_2 and to open the other: the result is a difference in the pressures P' and P''.

This is the principle of operation of the double flapper-and-nozzle hydraulic potentiometer which constitutes the first hydraulic stage of this servo valve:

$$[\ldots C \rightarrow (P_0' - P_0'') \ldots]$$

The valve spool is set in motion as a result of the action of the pressure difference $(P_0'-P_0'')$. This movement compresses one of the end springs and extends

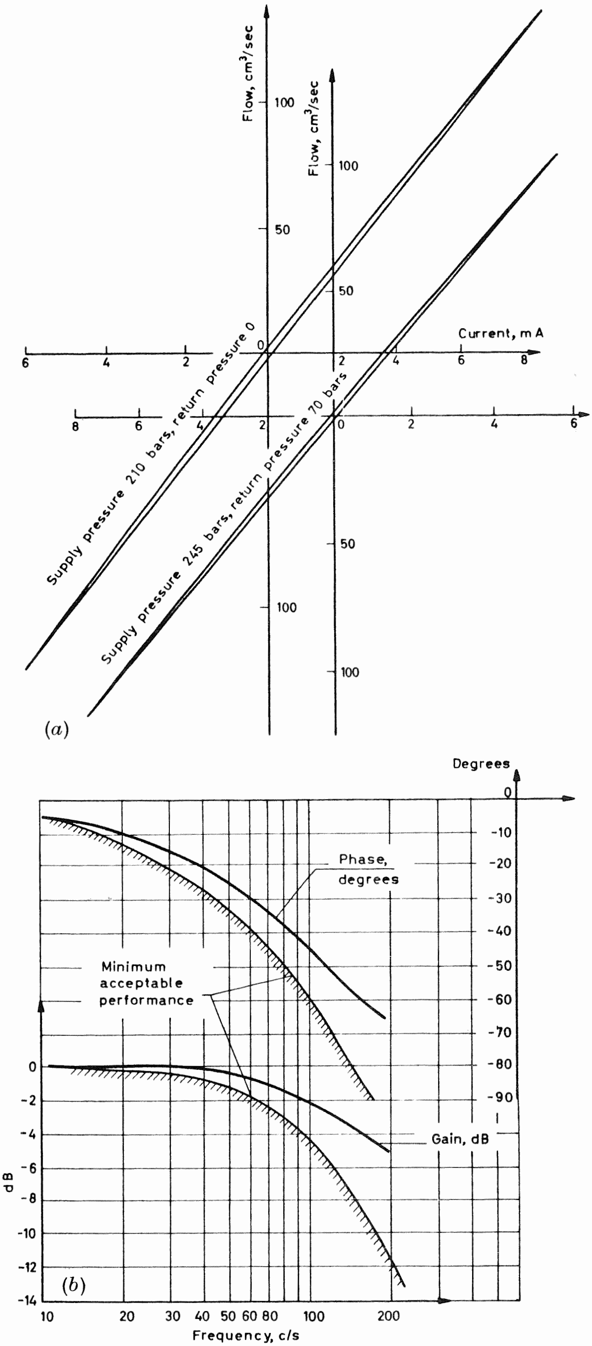


Figure 8.3. Servo valve type 30261: (*a*) static characteristics; (*b*) frequency response curves

Temperature, 20°C; oil temperature, 25°C; mean (quiescent) current, 4mA; $\Delta i = \pm 3$mA; $\Delta p = 210$ bars the other. Movement ceases when the force from the springs balances the pressure force. Therefore, a certain spool position, y, corresponds to a certain pressure difference $(P_0' - P_0'')$:

$$[\ldots (P_0' - P_0'') \rightarrow y \ldots]$$

The input current, therefore, controls the position of the valve spool.

Let us assume that the opening of the valve connects one side of a ram with the supply pressure and the other side with the reservoir. If the cross-sectional areas of the valve openings are proportional to y, they are also therefore proportional to the input current.

But the equation for the flow, Q, is

$$Q = Ks\sqrt{\Delta P}$$

so that, in order to provide a true flow servo control, the value of ΔP should be kept constant.

This is not true of the valve described nor of the great majority of existing valves, since ΔP varies with the supply pressure and with the load on the ram. (This point will be clarified in Section 8.2.3.)

We may conclude that the so-called flow servo valves do not give a flow proportional to the input unless the supply pressure is constant and the ram load is small. This is no real disadvantage, since perfect flow control is seldom required and can, if required, in any case be provided relatively easily by the addition of a supplementary loop.

8.2.2.2. *Performance*

Figure 8.3a gives the experimental static characteristics of a type 30261 servo valve. The curves show the variation of flow with input current for zero load and two values of the supply and return pressures. These curves are copied directly from a special test rig which operates continuously during increasing and decreasing values of the input current, and they clearly indicate the hysteresis and linearity of the servo valve.

Figure 8.3b gives the experimental dynamic characteristics of the same valve, i.e. its frequency response curves.

The performance is particularly good—note, for example, that the phase shift reaches 45° only in the neighbourhood of 100 c/s, which corresponds to an equivalent time constant of 1·6 msec.

Note—Many manufacturers give an approximate mathematical form of the transfer function for their servo valves. These first-, second- or third-order functions are useful when using an analogue computer to investigate a hydraulic problem. When applying the methods given in Chapter 6, however, it is easier to use the transfer locus such as that shown in *Figure 8.3b* (cf. e.g. Example 7 in Chapter 10).

Table 8.1 summarizes the performance of servo valve type 30261 and gives an indication of the order of magnitude of the performance figures for other serve valves listed later.

The speeds of response are sufficiently high for suitable servo valves to be found for present electrohydraulic designs, since at the operational frequencies the phase shift is only of the order of a few degrees. In addition, because of the specific form of the hydraulic transfer loci, a small phase shift at the natural frequency of the ram sometimes has a beneficial effect on the stability of the system.

On the other hand, research is required to improve the static performance, in particular to decrease the hysteresis and the null shift as a function of various parameters such as pressure, temperature, mean current, etc. The following

Table 8.1

Parameter	*AE30261*	*Other servo valves*				
Maximum pressure, kg/cm²	210 (3,000 lb/in.²)	150–300 (2,000–4,000 lb/in.²)				
Maximum flow, cm³/sec	150 (2 gal/min)	<400 (<5 gal/min)		400–2,000 (5–25gal/min)	>2,000 (>25 gal/min)	
Number of hydraulic stages	2	1	2	2	2	3
Electric power, W	0·1	0·5–5	0·05–1	0·1–2	0·5–5	0·05–2
Permanent internal leakage flow, cm³/sec	<10	1–10	3–20	6–30	>15	>30
Frequency at which phase shift reaches 90° c/s	200	100–300		50–250	20–60	40–100
Weight, kg	0 45 1	0·3–2 0·6–4	0·25–1 0·5–2	0·4–4 1–8	>2 >4	
Dead zone	<1 per cent	<1–5 per cent				
Hysteresis	<2 per cent	<5 per cent				
Gain, G	$\frac{\Delta G}{G} < 5$ per cent	constant to ± 10 per cent except perhaps near zero and maxima				
Maximum operating temperature	ca. 200°C	ca. 200°C				
Maximum acceleration that can be withstood	100 g	ca. 50–100 g				
Maximum vibration that can be withstood	500 c/s 50 g	up to 500 or 1,000 c/s for amplitudes giving accelerations of up to 30 or 50 g				
Null shift	<7 per cent (temperature) <12 per cent (acceleration)	5–20 per cent for the extreme temperatures and accelerations quoted above				

Chapter shows how this null shift affects the performance of the chain.

Summarizing then, the most important performance, i.e. the dynamic permance, is not the most difficult to achieve nor the most urgently in need of improvement.

8.2.2.3. *Efficiency and output power*

Engineers often enquire about the 'efficiency' of a servo valve when it does not really have any significance. The efficiency is defined as the ratio of the output power to the input power and it is always possible to select a servo valve which has a maximum opening sufficiently large for the loss of head caused by the servo valve to be negligible in comparison with the pressure necessary to move the load.

But a servo valve is not meant to operate fully open—its function is to adjust a flow or orifice accurately in response to a given command. It could be considered as reducing the available energy in order to supply only that quantity which it was ordered to supply.

The variation of the efficiency with orifice area, for a given application, is an important characteristic of the electrovalve, but not the servo valve.

On the other hand, an evaluation is often required of the power supplied by a particular servo valve.

Let the flow through a servo valve with a supply pressure P_1 be Q_1 under no load and let H_1 be the *arbitrary* reference power P_1Q_1.

Now suppose that the servo valve supplies fluid to the chamber of a loaded ram. Let P_L be the pressure necessary to move the load and let $P_A = P_1 - P_L$ be the pressure available to cause the flow. The new flow is, therefore

$$Q = Q_1\sqrt{\frac{P_A}{P_1}}$$

and the new power is

$$H = Q_1\sqrt{\frac{P_A}{P_1}}\,.\,P_L$$

$$= H_1\sqrt{\frac{P_A}{P_1}}\left(1-\frac{P_A}{P_1}\right)$$

H is obviously zero for $P_L = 0$ (no load) and for $P_L = P_1$ (no flow) and has a maximum value, H_{max}, when $P_A = P_1/3$:

$$\therefore\ H_{\text{max.}} = \sqrt{\tfrac{1}{3}}(1-\tfrac{1}{3})H_1 = 0{\cdot}385\,H_1$$

This result is the origin of the advice given frequently but without real justification, to design a hydraulic system so that $\frac{2}{3}$ of the pressure is used for the load and $\frac{1}{3}$ to give the flow through the valve.

8.2.2.4. *Gain*

The preceding calculation gives a rapid evaluation of the maximum power gain of a servo valve.

As an example, consider the servo valve type 30261 described above. It is fully open when the input current is about 8 mA, which corresponds to an input power of 0·1W since the resistance is 1,500 Ω.

With a supply pressure of 210 bars (3,000 lb/in.^{2}), the flow is 150 cm^3/sec (2 gal/min) under no load, so that the maximum power output is $0{\cdot}385 \times 150 \times 210 \times 10^{-1} = 1{,}200$ W. *Its maximum power gain is therefore 12,000.*

Other servo valves have even higher values of the power gain. For example, servo valve AE 30265 has all characteristics identical with those of 30261 except its maximum flow, which is 350 cm^3/sec (4·6 gal/min), giving a gain of 28,000. Servo valve AE 30277, with very similar input characteristics, gives a flow of 900 cm^3/sec (12 gal/min) under a pressure of 300 bars (4,300 lb/in.^{2})—a gain in excess of 100,000.

8.2.2.5. *Technical considerations*

Servo valves have to be manufactured to an accuracy of 1 μ. It will be shown that this accuracy is needed because of size considerations.

First note that in order to keep the size and weight to reasonable proportions, the pressures used are high, generally of the order of 200 bars (3,000 lb/in.^{2}) or more.

Also, to avoid excessive hysteresis, servo valves do not contain any elastic sealing elements. The fluid is restrained from leaking by the quality of the fit between adjacent moving parts.

We will calculate the average speed of the fluid through a valve orifice under a pressure difference of 200 bars (3,000 lb /in.^{2}). Putting $\xi = 1{\cdot}8$ and $\rho = 0{\cdot}87$ in eqn. (4′) of Chapter 1, we get $V = 160$ m/sec. Thus for a 1 mm (0·04 in.) diam. hole, the flow is 130 cm^3/sec (1·7 gal/min).

In servo valve 30261, the total allowable internal leakage flow, i.e. the sum of the hydraulic potentiometer flows and the real leakage flow past the spool, is 10 cm^3/sec (0·132 gal/min). In practice, the internal leakage flow is about 5 cm^3/sec (0·066 gal/min), i.e. about 4 per cent of the flow through the 1 mm hole. This is equivalent to a hole of 0·2 mm (0·008 in.) diam. pierced between the high- and low-pressure areas. About half of this internal flow is due to the clearance between the spool and the sleeve and half is the hydraulic potentiometer flow.

It is not surprising, therefore, that the clearance between spool and sleeve is $3 \pm 0{\cdot}5\ \mu$ and that the diameter of the fixed orifices O_1 (see *Figure 8.2*) is about 0·11 mm (0·0043 in.).

But there is worse to come! The pressure differences between the supply and the first chamber of the ram, and between the second chamber of the ram and the sump, are both of the order of 100 bars (1,400 lb/in.^{2}). The speed of the fluid is therefore about 110 m/sec (4,000 in./sec) so that, for a maximum flow of 150 cm^3/sec (2 gal/min), an area of 1·36 mm^2 ($2{\cdot}1 \times 10^{-3}$ in.^{2}) is required. If this flow area is caused by uncovering two slots, each 2 mm (0·081 in.) wide, the maximum opening will be about 0·34 mm (0·0134 in.).

Now, if the non-linearity at the moment when the flow changes direction, i.e.

when one slot closes and the other opens, is to be less than 1 per cent of the maximum flow, the ends of the four opening slots must be positioned to an accuracy of $0{\cdot}34/100 = 3{\cdot}4\ \mu$. In servo valve 30262, which has the same characteristics and a maximum flow of 40 cm^3/sec (0·53 gal/min), a linearity error of 2 per cent corresponds to an accuracy of 2 μ (less than 0·0001 in.).

8.2.3. VARIOUS TYPES OF SERVO VALVE

8.2.3.1. *Why are there so many types?*

The fact that so many different types of servo valves are available is partly due to commercial reasons: competition, patents, contracts. Another reason is the novelty of this component. Its comparatively recent introduction and rapid development have resulted in a variety of designs, some of which will not last—several principles which initially seemed attractive have already been rejected. The remaining reasons are purely technical: servo valves are needed to meet different requirements.

The different *functions* which may be required of a servo valve are

(*i*) To provide a flow proportional to the input current but where the constant of proportionality may vary with the supply and return pressures and also with the resistance of the receiver. Hence, we would use a servo valve in which the *position* of the spool is controlled by input current, as is the case in the servo valve described;

(*ii*) To give a flow which is exactly proportional to the input current. We would therefore need a *true* flow-control servo valve;

(*iii*) To provide an output pressure proportional to the input current. We would thus need a *pressure-control servo valve*;

(*iv*) To give a more complex function, which may be achieved by a *partial pressure control* servo added to a normal servo valve;

(*v*) To give a flow proportional to the *integral of the input current*. We would therefore use an acceleration switching servo valve where, as its name implies, the acceleration of the ram and not, as before, its speed, is proportional to the input current.

The different *energy levels* which may be required also vary considerably. Servo valves may have maximum flows between, say, 10 and 40,000 cm^3/sec (ca. 0·12–600 gal/min) under pressures ranging from 40 to 400 bars (ca. 600–6,000 lb/in.^{2}) while the allowable electrical input power may vary between $\frac{1}{20}$ and 20 W. The servo flow must be only a small fraction of the total flow; so servo valves may have one, two or three hydraulic stages.

The different *quality criteria* listed in Section 8.2.4 will affect the design according to the way in which the servo valve is to be used.

This explains the variety of technological, technical and analytical aspects which must be considered in the design of each element of a servo valve.

The electric motor—The electric motor produces either a torque (*Figure 8.4a*) or a force (*Figure 8.4b*) proportional to the current applied to it. It is usually of the direct-current, permanent magnet type, often with a push–pull input; sometimes it is a loudspeaker-type coil.

There is a trend towards low-power electric motors which have the advantages of reducing weight, size and time constant of the electrical stage together with simplification of the electric amplifier, but which usually require a hydraulic preamplifier in the servo valve.

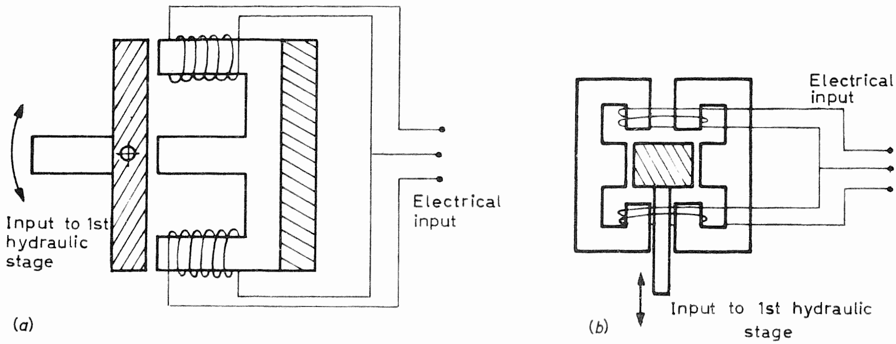


Figure 8.4

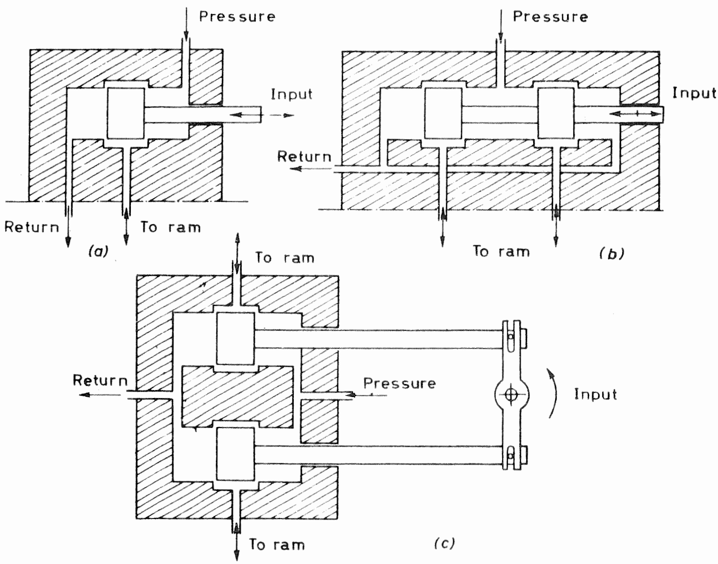


Figure 8.5

They can be divided into two categories:

(*i*) *Dry* motors, which are not in direct contact with the hydraulic fluid; these introduce the difficult design problem of effecting a frictionless seal for the transmission.

(*ii*) *Wet* motors where the windings are immersed in the hydraulic fluid. These can be subdivided into

(*a*) those with *circulating fluid*, whose performance varies with time owing to the attraction and accumulation of metallic particles which are suspended in the fluid;

(*b*) those in *still fluid* where the liquid surrounding the windings is not circulated and therefore causes no contamination.

Wet motors are simpler than dry motors and have better thermal equilibrium.

The main hydraulic stage—This is usually either a 3- or 4-way spool valve, depending on whether it supplies a differential- or equal-area ram (*Figures 8.5a* and *b*). The 4-way valve may be replaced by two half valves (*Figure 8.5c*).

Most valves have square-edged spools and correspondingly recessed sleeves with 90° edges. This form introduces unwanted hydraulic forces (see Chapter 10, Example 4) but is relatively easy to produce. The manufacture of balanced valves with profiled spools to the tolerances indicated in Section 8.2.2.5 is almost impossible.

The first hydraulic stage—Owing to the low power of the electric motor and the

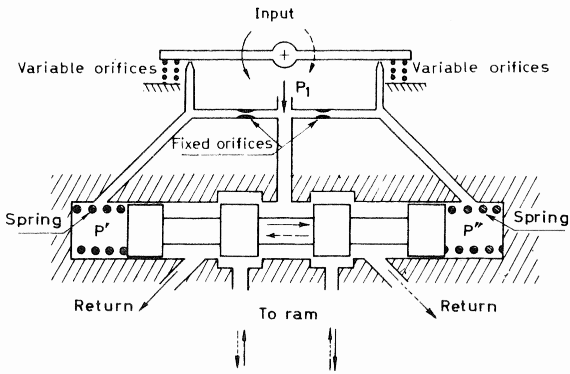


Figure 8.6

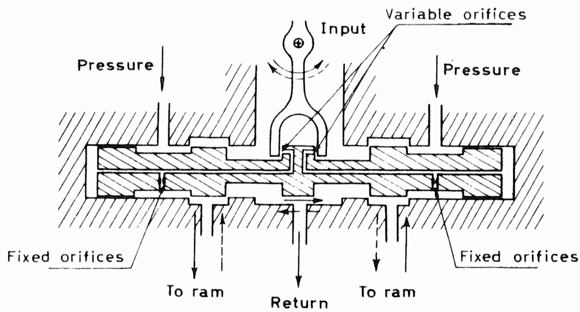


Figure 8.7

hydraulic forces introduced by the square-edged main valve, a hydraulic amplifying stage is required.

The first stage may be any of the following types: spool, sliding plate, rotary or movable jet. More often, however, it is the single or double hydraulic potentiometer type with flapper-and-nozzle valve.

The first hydraulic stage controls the main valve by applying two pressures to the end faces of the main spool. Unless special precautions are taken, the result is an acceleration valve. But the spool can be position-controlled by various means, such as centring springs providing open-loop control, as in the case of the valve described above (*Figure 8.6*); a follow-up process, so that displacement of the spool is the same as that of the first-stage moving member (*Figure 8.7*); or else, a feedback force upon the first-stage moving member, so that the spool movement causes the first stage to be restored at neutral.

In some designs, either or both hydraulic stages may be supplied with electrical position detectors, so that the internal servo control is accomplished by an external electronic circuit.

An extensive review of servo valves has been given by Kinney and Weiss[1]. A few examples of each type of valve will be given under the general classification used by these authors.

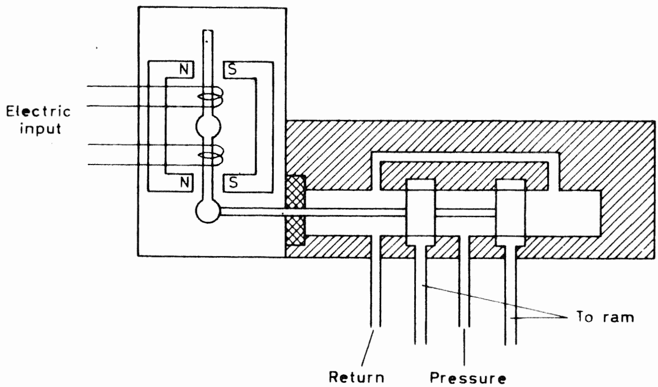


Figure 8.8. Lear 5214 spool valve

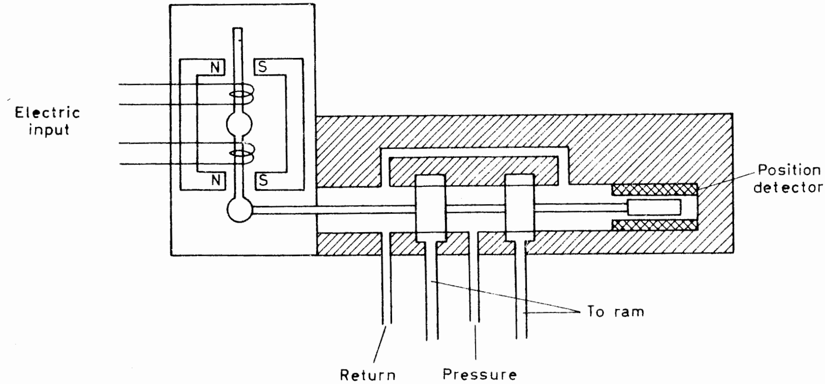
8.2.3.2. *One-stage servo valves*

The connecting together of an electric motor and a hydraulic valve may constitute a servo valve, and this arrangement is employed when only moderate performance is required. Apart from inferior performance, there is no neat solution to the problems of sealing and hysteresis (Section 4.4).

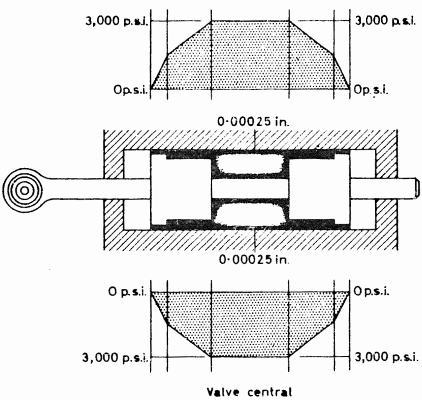
Normally, the term servo valve is reserved for compact components where the motor and the valve have been designed to function together as a unit.

One-stage servo valves are simple, their internal leakage flow is small since there is no servo flow, but their gain is limited by the presence of uncompensated hydraulic forces. They are used either for low pressures, small flows or with relatively powerful torque motors which have correspondingly large time constants, seriously affecting the performance of the servo valve as a whole. Examples of this type of valve are:

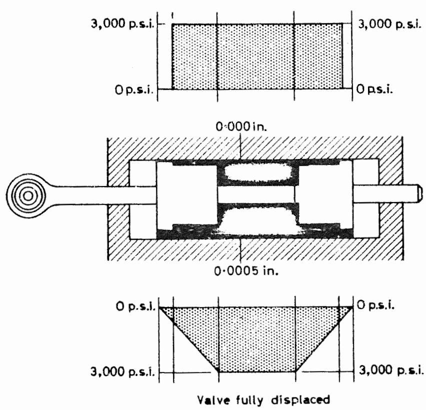
Lear 5214 (*Figure 8.8*)—Spool valve with permanent magnet type dry torque



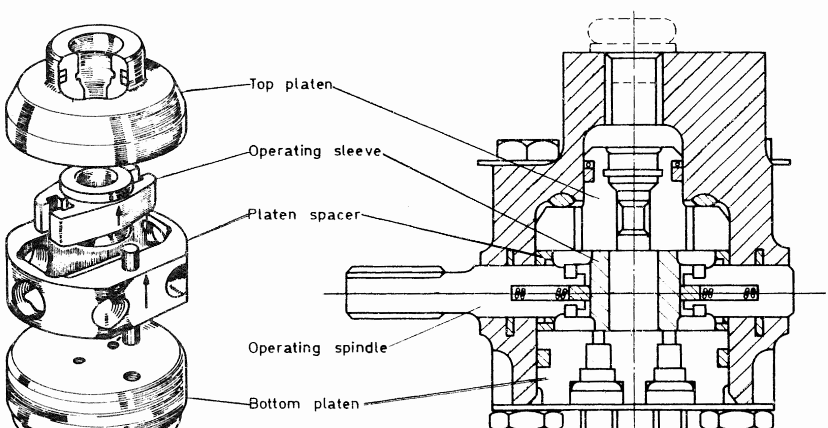
(*a*)



(*b*)



(*c*)



(*d*)

Figure 8.9. (*a*) Minneapolis Honeywell Series VJ valve; (*b*)–(*d*) Fairey flat face servo valve

motor; maximum flow 60 cm^3/sec at 210 bars (3,000 lb/in.^{3}); input power 0·6W (0·8 gal/min); leakage flow 3 cm^3/sec.

Minneapolis Honeywell Series VJ (*Figure 8.9a*). Spool valve with permanent magnet, non-circulating fluid-type torque motor; inductance position detector on the valve spool; maximum flow 60–250 cm^3/sec (0·8–33 gal/min) at 70 bars (1,000 lb/in.^{2}); input power 4W; leakage flow 1·5 cm^3/sec.
American Measurement (rotary disc valve).
Midwestern Instruments (two half valves).
Hydraulic Research; *Oilgear*; *Bendix Hamilton*; and two French designs: *S.A.M.M.* (rotary valve) and *Air Equipement* type *30072*.

Another example is the *Fairey servo valve* which has a plane slider instead of an axially symmetrical spindle. An interesting arrangement enables metal-to-metal contact between fixed and moving parts to be avoided, so that it can be driven directly by a torque motor and only a single stage is required. Hydrodynamic recentring is accomplished by converging laminar passages (*Figure 8.9b, c*) which are easier to manufacture on a plane slider than on an axially symmetric spindle.

The control valve (*Figure 8.9d*) consists of a top platen (supply), a bottom platen (delivery), a plane sleeve and a spacer maintaining a clearance of 10 μ between the faces. The faces of the platens and the sleeve are optically flat, and the end faces of the sleeve have steps of approximately 2 μ which give the hydraulic recentring effect clearly shown in *Figure 8.9b, c*.

8.2.3.3. *Two-stage servo valves: spool + spool*

In this type of servo valve, the first hydraulic stage consists of a spool-sleeve assembly which provides a flow proportional to the input signal, which in turn actuates the second stage, also a spool-sleeve assembly.

The first-stage spool is difficult to manufacture, since it is very small, but against this there is no constant wastage of internal servo flow. This arrangement is therefore most suitable for very large flows and in cases where the permanent flow must be kept to a minimum.

In addition to the manufacturing difficulty, there are two other problems: (*a*) protection of the first stage from contamination and clogging; (*b*) provision of a satisfactory feedback between the two stages. A feedback element is necessary because the first stage provides a *flow* and is therefore an *integrator*.

The method of establishing this feedback effectively subdivides this type of servo valve into two further categories:

(*i*) those with *position feedback*.
(*ii*) those with *mechanical feedback*.

In the first category, the sleeve of the first stage may be controlled by the spool of the second stage by means of a lever (*Figure 8.10*).

Examples of this type of valve are the *Vickers E 110894* (flow exceeds 1,000 cm^3/sec (13 gal/min) at 210 bars (3,000 lb/in.^{2}); input power, 10W; weight 7 kg = 15 lb) and the *Raytheon* valves 090688–0918, with smaller flows but having exceedingly low internal leakage (of the order of 1 cm^3/sec).

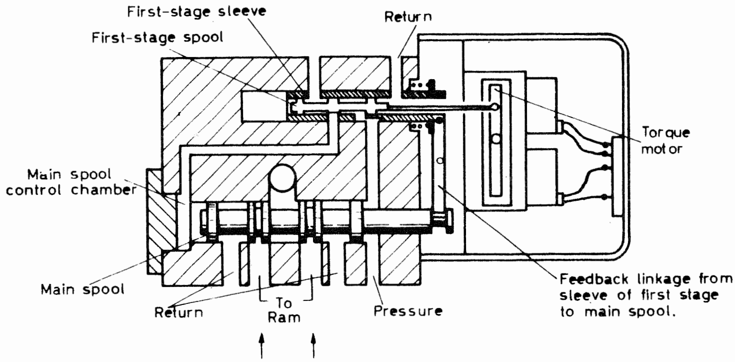


Figure 8.10. Vickers valve

The sleeve of the first stage may also be connected directly to the main spool (*Figure 8.11*). Examples of this type are the *Hagan* models *80* and *600*, industrial valves with high maximum flows in excess of 40,000 cm^3/sec (500 gal/min) under 140 bars (2,000 lb/in.^{2}). These are two of the very rare valves in which the spool shape has been designed to minimize the dynamic hydraulic forces (cf. Section 4.3.4.3).

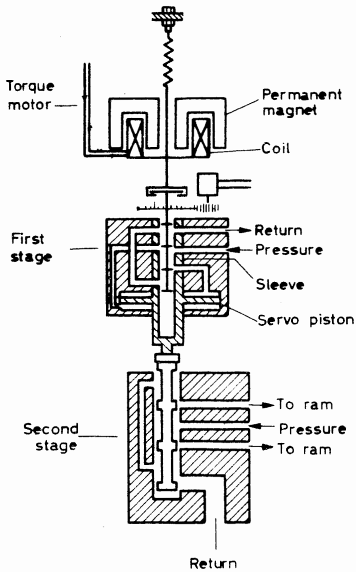


Figure 8.11. Hagan valve

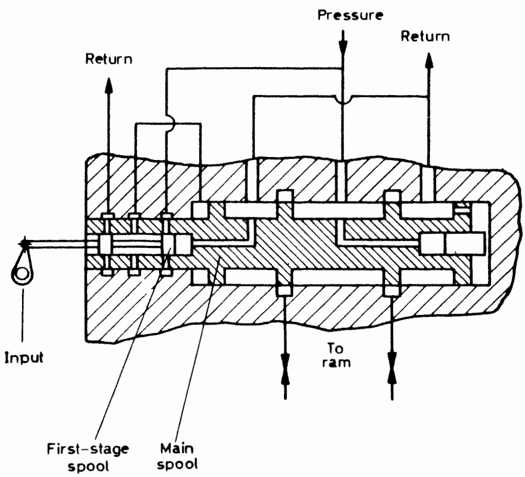


Figure 8.12. Sperry valve

Lastly, the sleeve of the first stage may itself constitute the main spool (*Figure 8.12*). Examples of this type are the *Sperry* valves, sold without torque motor.

In the second category, the position of the main spool is measured and transformed into a force which is applied to the spool of the first stage in opposition to the main control force. This can be done *mechanically*, for example by a spring connecting the ends of the first- and second-stage spools.

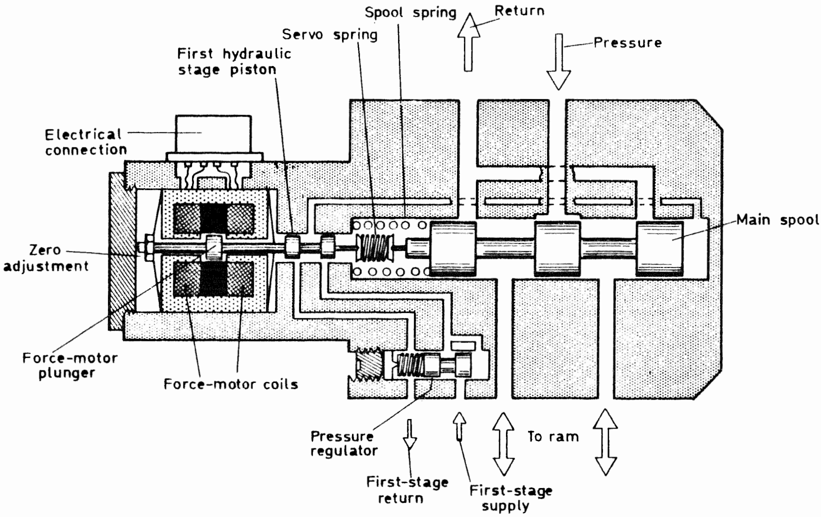


Figure 8.13. Sanders valve SV522

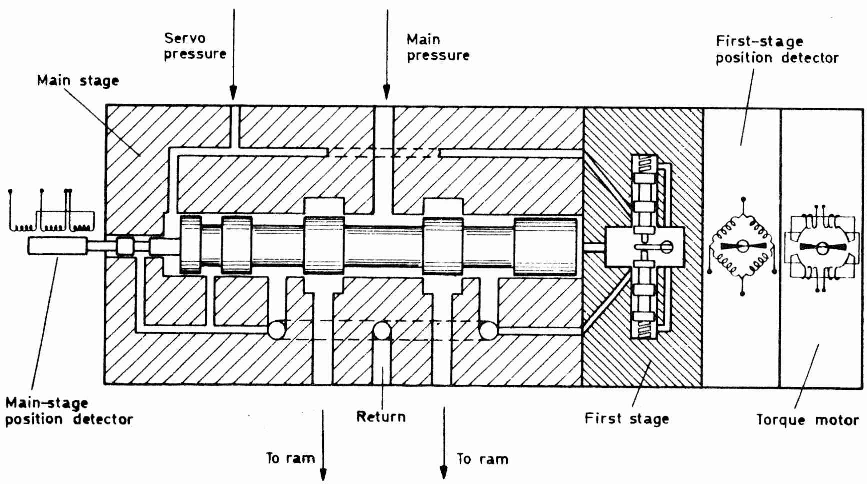


Figure 8.14. Sigma Keelavite valve DT2 504b

Since the first-stage spool undergoes only small displacements, the force exerted on it by the spring is effectively only a fraction of the position of the main spool; at the same time, the position of the latter is practically unaffected by the force of the spring which is very small in comparison with the control force acting on the main spool.

An example of this type of valve is the *Sanders SV 522* (maximum flow 10,000 cm^3/sec or 130 gal/min, under 210 bars, or 3,000 lb/in.^{2}) shown in *Figure 8.13.*

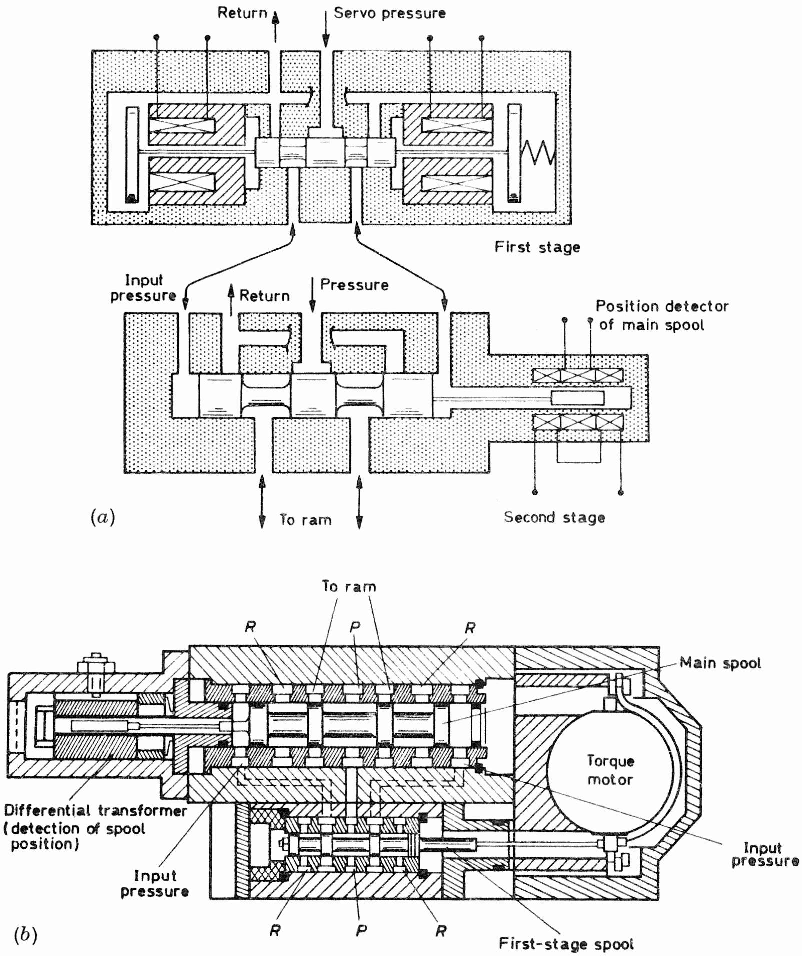


Figure 8.15. (*a*) Bertea Products valve; (*b*) Hydraulic Controls valve D52

The measurement of the position of the main spool can also be made *electrically* and likewise the subtraction, ahead of the torque motor. An example of this is the *Sigma Keelavite DT 2504 G* valve (*Figure 8.14*) in which the first-stage spool is also controlled electrically, thus giving exceptionally good accuracy. Similar examples are the *Bertea Products* valve used in the Nike Zeus rocket (*Figure 8.15a*) and the *Hydraulic Control D 52* valve (*Figure 8.15b*).

8.2.3.4. *Two-stage servo valves: flapper-and-nozzle valve + spool*

In this type of servo valve, the first hydraulic stage consists of a hydraulic potentiometer with flapper-and-nozzle valve, single or double, which provides a *pressure* (or pressure difference) proportional to the input signal and is used to actuate the second stage, consisting of a spool–sleeve assembly.

This group includes the valve described in Section 8.2.2. It is the most common arrangement and has been used by all the manufacturers. The performances obtained by the various firms are good and remarkably similar.

For flows between 25 and 1,000 cm^3/sec (0·3 and 13 gal/min) under 200–300 bars (3,000–4,200 lb/in.^{2}), the electrical input powers are of the order of 0·05–0·2W, the cut-off frequencies at 90°, 100–200 c/s, the internal leakage flows 4–20 cm^3/sec and the weights about 500 g (1 lb).

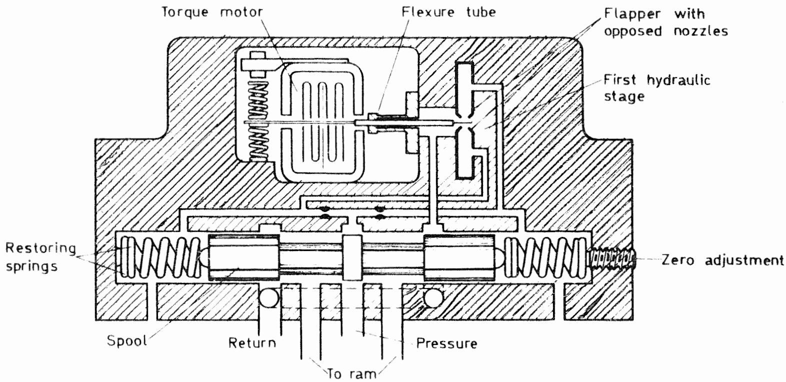


Figure 8.16. Moog valve 21

This large group of valves can be subdivided into servo valves with

(*i*) *no internal feedback* (since the input to the second stage is a pressure, it is possible, as seen in Section 8.2.2 and contrary to the case of the spool + spool valves, to fix the position of the main spool by stiff springs)

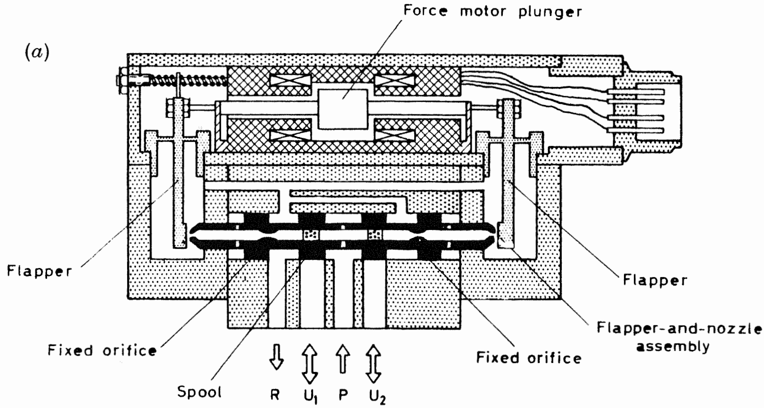
(*ii*) *position feedback*

(*iii*) *mechanical feedback*

Servo valves with no internal feedback—So many models of this type are available that it is impossible to enumerate them. The flapper plate may be floating or pivoted, the nozzles parallel or opposed, and the motor may be dry or wet.

Among the better known models are the *Hydraulic Research* series *20–22*, *Moog* series *21* and *22*, *Vickers*, *Dalmo*, the *Bendix* design described and, in France, the *Messier*. *Figure 8.16* shows the layout of the Moog 21 which has a dry motor and opposed variable orifices.

Servo valves with position feedback—In these valves, the variable orifices are mounted on the main spool and the first stage is known as a hydraulic follow-up system.



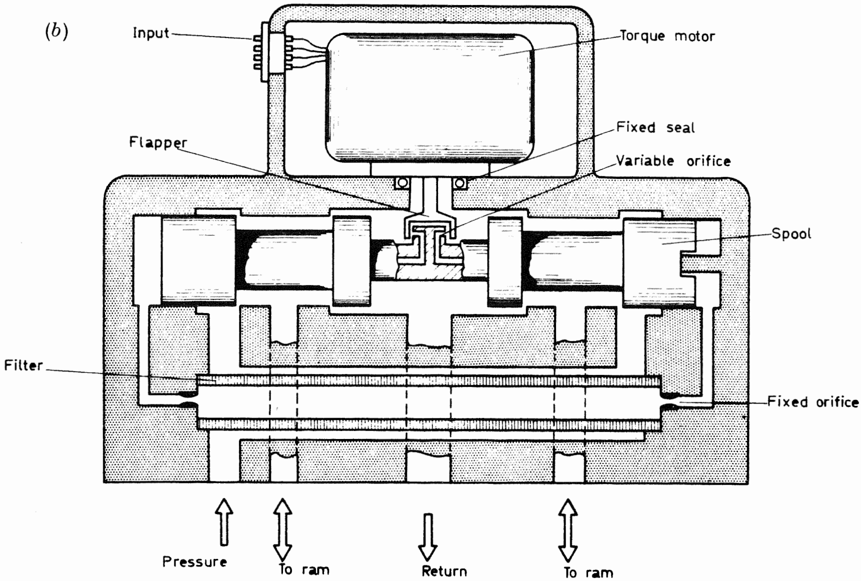


Figure 8.17. (*a*) Pegasus valve; (*b*) Bendix Hamilton valve

Standard designs are those such as the *Pegasus* (*Figure 8.17a*) or *Bendix Hamilton* valves (*Figure 8.17b*).

Hydraulic centring servo valves may also be included in this category. The fixed orifices of the hydraulic potentiometer which constitutes the first stage are here replaced by tapered feedback slots cut in the main spool, so that the orifices are a function of the *position* of this *spool*. Two valves based on this principle are the English *G.E.L. 144/16* (*Figure 8.18*) and the French *S.O.M.* (*Figure 8.19*).

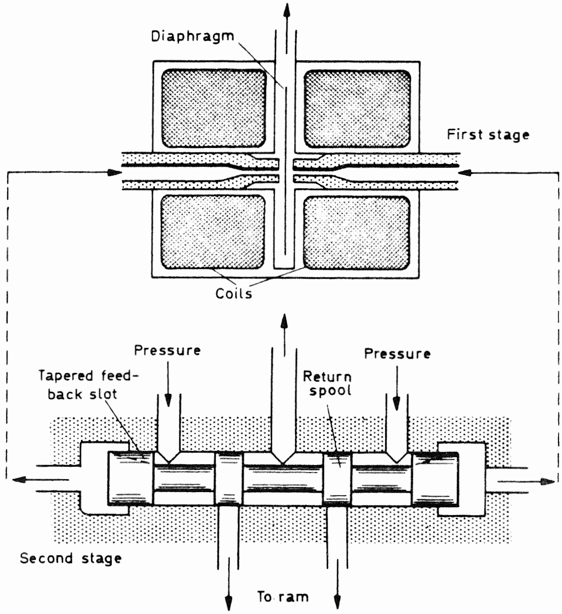


Figure 8.18. G.E.L. valve 144/16

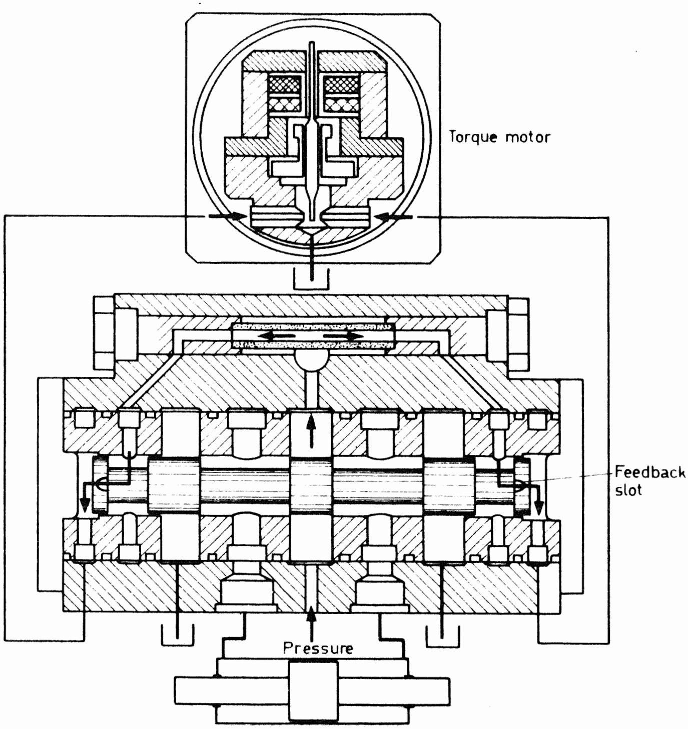


Figure 8.19. S.O.M. valve

Servo valves with mechanical feedback—As we have seen already for the spool + spool configuration, valves of this type measure the position of the main spool and transform this measurement into a force which is applied to the first stage.

It has also been shown that this type of feedback may be achieved by a simple spring.

Rapid development of this type of valve is taking place at present, since this simple configuration is capable of giving large improvements of static performance.

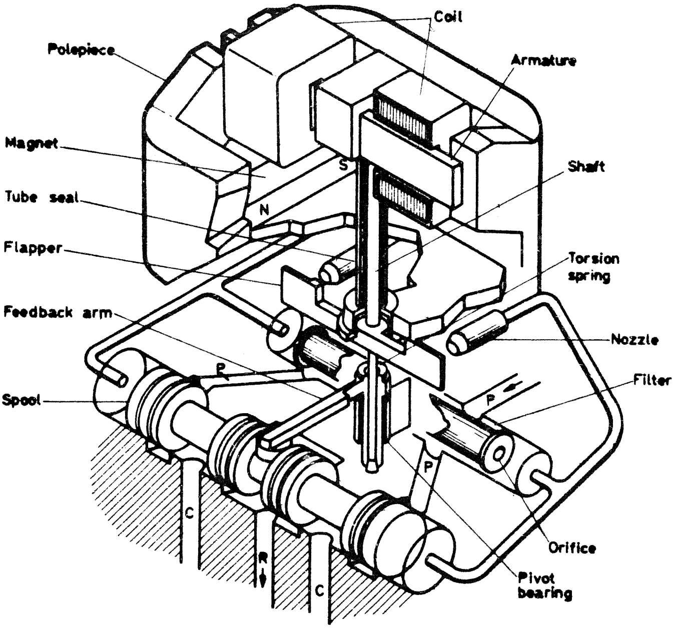


Figure 8.20. Bendix Pacific valve

Again it is impossible to quote all the designs available, but it is worth noting that the internal feedback spring may be

a torsion shaft or tube: *Bendix Pacific* valve (*Figure 8.20*).

a metal strip: *Moog 30* and *35* (*Figure 8.21a*) and *Hydraulic Research 25–26* (*Figure 8.21b*).

A coiled spring: *Sanders 219* and *324* (*Figure 8.22*), *Weston* (*Figure 8.23*) and the industrial *Air Equipement* valves *31022–31025* (*Figure 8.24*).

8.2.3.5. *Particular designs of servo valves*

There are many other designs of servo valves. The following summary gives an idea of the construction of the commoner types.

(*1*) *The movable-jet valve*—In this type of valve a movable nozzle is mounted on the torque motor flapper plate and supplied with fluid under pressure. It directs a jet of fluid towards two fixed orifices, its direction being controlled by the input current. The direction of the jet relative to the two orifices controls the relative pressures in them which, in turn, control the movement of the main spool.

The main advantage of this type of valve is its freedom from clogging and contamination problems.

A typical movable-jet valve is the *Raymond Atchley* valve (*Figure 8.25*). For

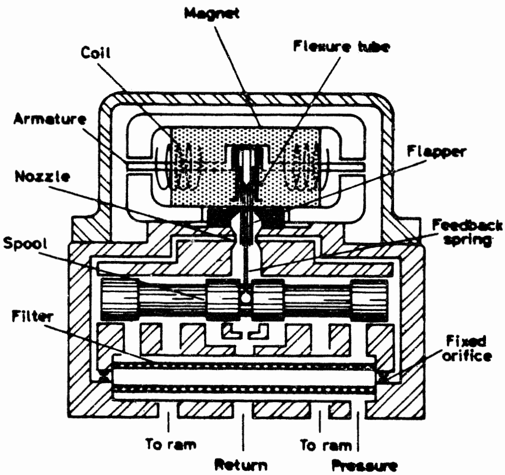


Figure 8.21a. Moog valve 30

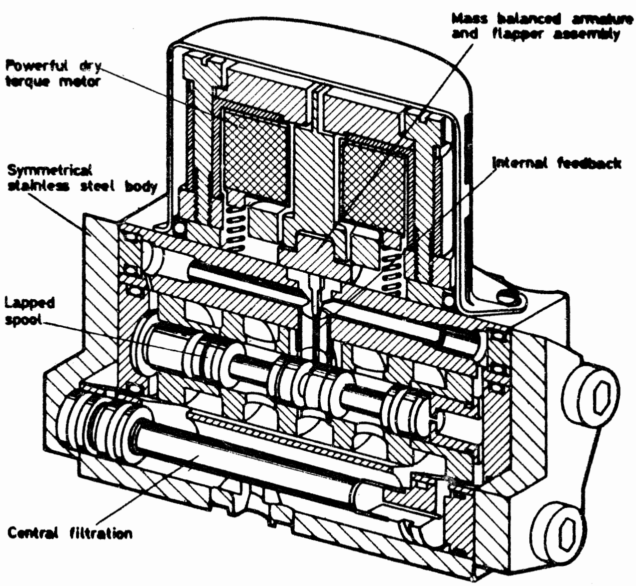


Figure 8.21b. Hydraulic Research valve 26

flows of 300–3,000 cm^3/sec, its performance is comparable to those of other designs of servo valves.

For reasons of reliability, some valves have two first stages in parallel, with an electrically operated spool-selector valve. The *Messier Cr 85205* is a valve of this type.

(*2*) *Asymmetrical valves* in which a null shift in a given direction takes place when the electrical input is switched off. Examples are the *Hydraulic Research* and *Air Equipement 30272* valves.

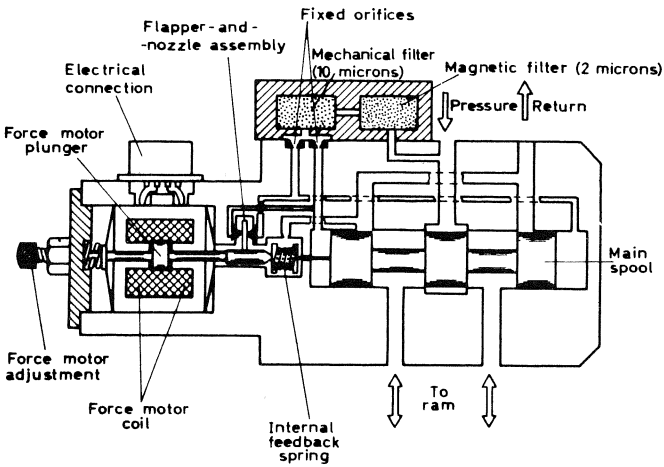


Figure 8.22. Sanders valve 324

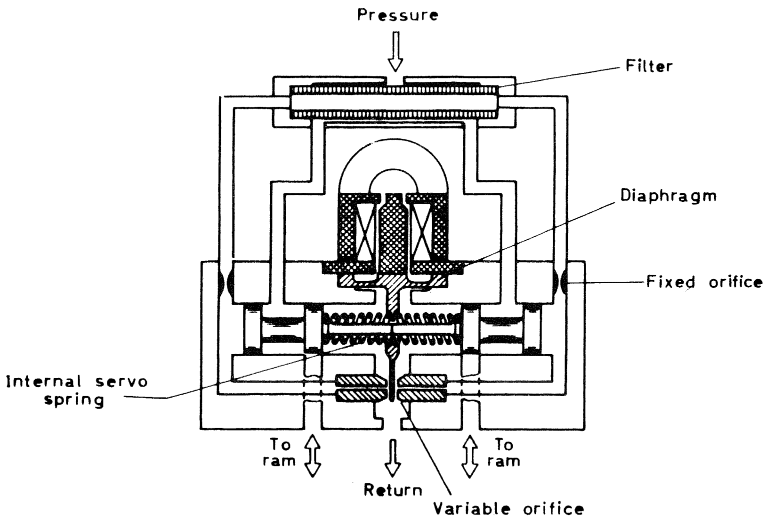


Figure 8.23. Western Hydraulics valve

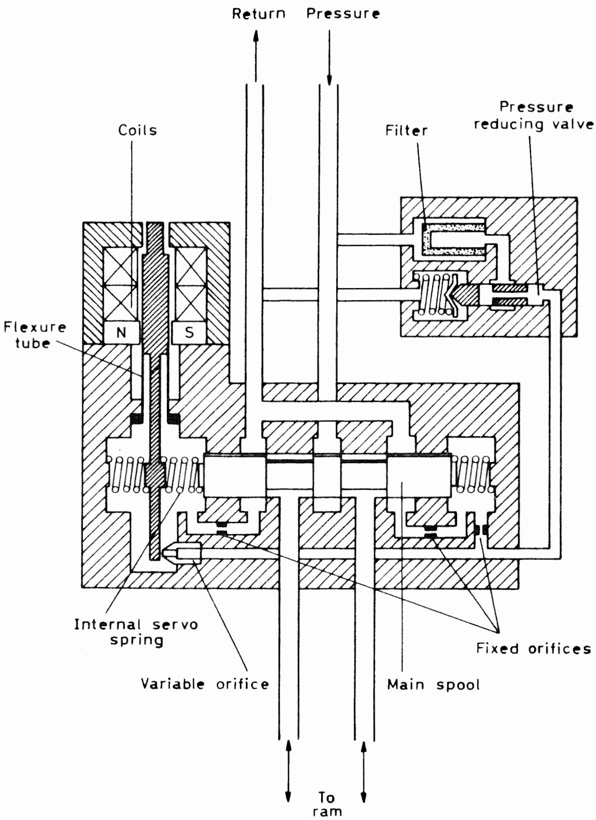


Figure 8.24. Air Equipement valves 31022–31025

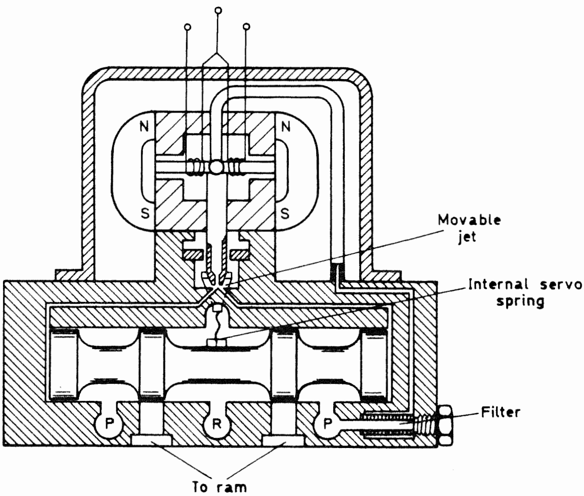


Figure 8.25. Raymond Atchley valve

(*3*) *Double-input valves* in which a mechanical input may be added to the electrical one or may even replace it completely. This type of valve has been developed to combine the actions of a human pilot with the automatic pilot of high-speed aircraft. The mechanical input is made either, as in the case of one valve having a hydraulic potentiometer and spool with springs, at the level of the second-stage springs or, sometimes, at the level of the first stage. Examples are the *Moog 10*, *Hydraulic Research Hydromat*, *Air Equipement 30550*.

(*4*) *Integrated valves*—These are designed in conjunction with a ram and receive its position signal mechanically. They do not, therefore, require any electrical feedback chain. Many valves of this type are manufactured by *Hydraulic Research*.

(*5*) *Valves with more than two stages* are designed for very large flows of the order of 3,000–30,000 cm^3/sec (40–400 gal/min). An example is the *Hydraulic Research Model 24* (*Figure 8.26*).

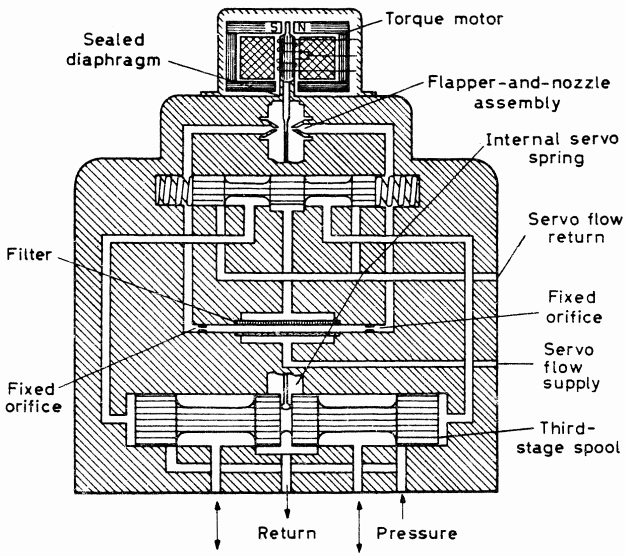


Figure 8.26. Hydraulic Research valve 24

8.2.3.6. *Particular functions of servo valves*

Despite the number of designs already quoted, we have still only considered servo valves whose function is to vary an opening in response to an electric command, somewhat incorrectly called *flow servo valves*.

There are servo valves which perform different functions, and the following are some examples.

(*1*) *True flow servo valves*—An example is the *Pesco* valve (*Figure 8.27*).

(*2*) *Pressure servo valves*

(*3*) *Valves with secondary feedback*—Since the most common type of secondary

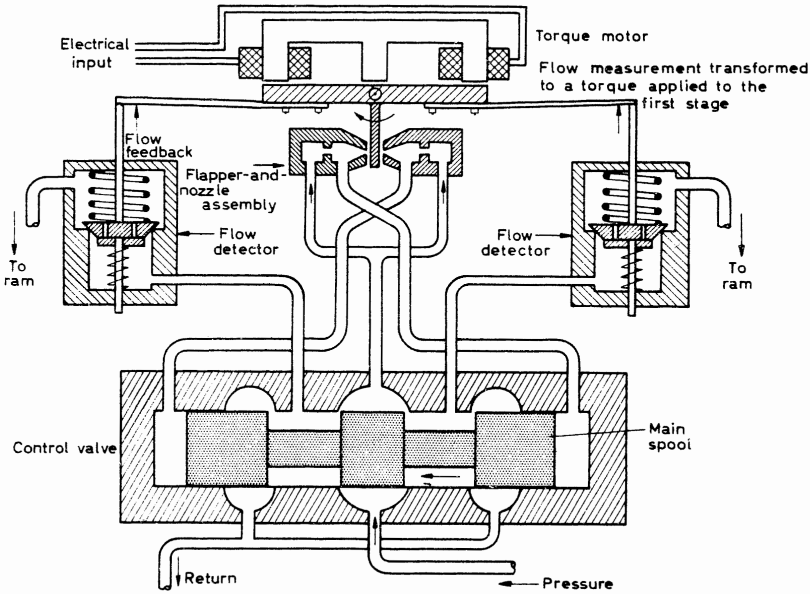


Figure 8.27. Pesco servo valve

feedback is that of pressure, there is often some confusion between these two types of valves.

A *pressure valve* is one whose function is to supply a receiver with flow at a given pressure, or pressure difference, independent of the magnitude of the flow absorbed by the receiver.

A *valve with secondary feedback* is a flow valve in which certain parameters, such as pressure, are detected and fed back into the first or second stage. This is done in order to improve the stability of the servo chain of which the valve is a part, but often at the expense of a zero error.

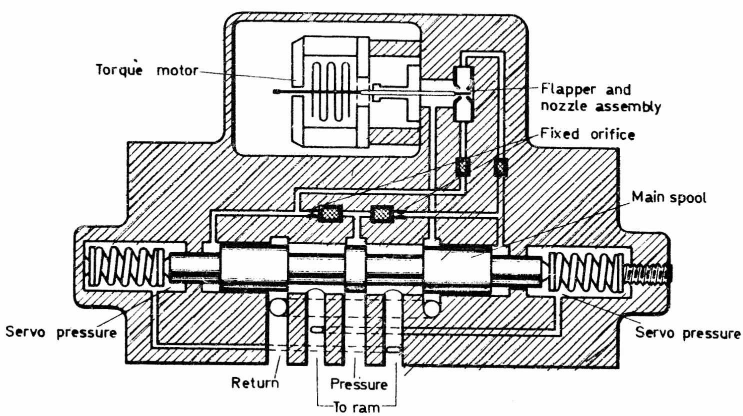


Figure 8.28. Moog pressure valve

A pressure valve can be formed from a flow valve (of the flapper-and-nozzle + spool type) by replacing the positioning springs at the ends of the spool by additional shoulders on the spool, thus allowing the output pressure to be applied to the spool. This is illustrated by the similarity between the *Moog* series 15 pressure valve (*Figure 8.28*) and the *Moog* series 21 flow valve (*Figure 8.16*).

This similarity also exists between the *Air Equipement 30262 and 30361 pressure valves* (*Figure 8.29*) and the *Air Equipement Bendix* flow valve (*Figure 8.1*), and between the *Messier A 25556 pressure valve* (*Figure 8.30a*) and the *Messier 25565 flow valve* (*Figure 8.30b*).

Also in this category are the original designs of *Ferranti* (movable-jet pressure valve) and *Cadillac* (pressure valve with two separate half valves, *Figure 8.30c*).

In the secondary feedback category, there are as many servo valves as there are problems to be solved. The *Moog* and *Hydraulic Research* companies in

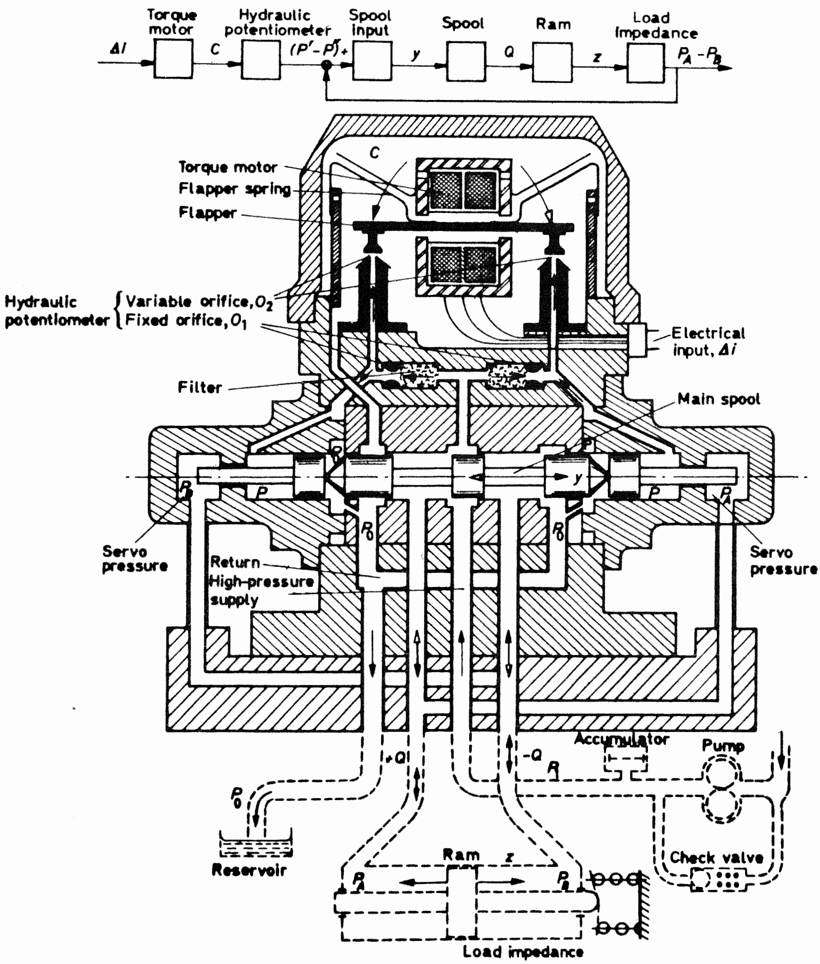


Figure 8.29. Four-way pressure servo valve (Air Equipement 30361)

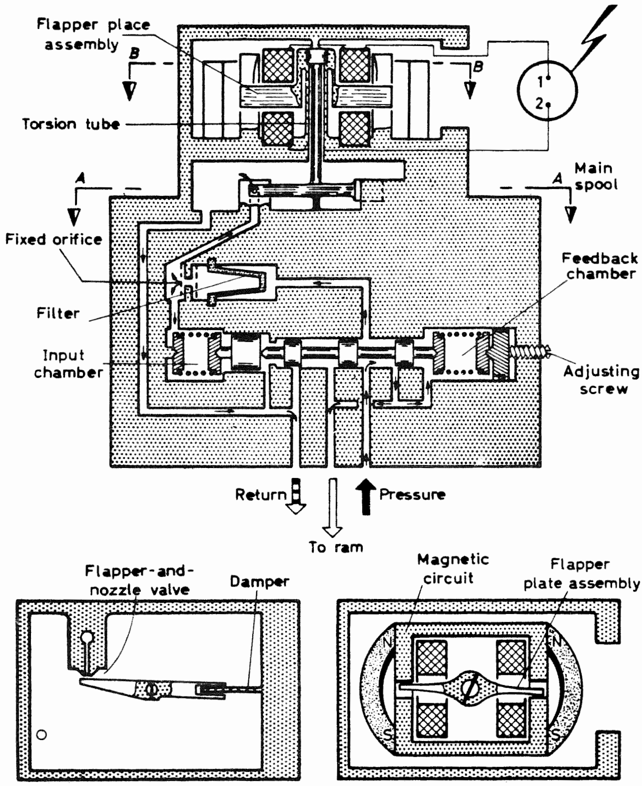


Figure 8.30. (*a*) Messier pressure valve A25556

particular have been active in this field. *Figure 8.31* shows some *Moog* pressure feedback valves. They all consist of a potentiometer and spool valve, but with internal feedback (*Figure 8.31a*), with simple pressure feedback (*Figure 8.31b*): as shown in Section 7.7.2.3, this introduces a static error. In *Figure 8.31c*, the pressure feedback is supplied with a high pass filter (dynamic pressure feedback) which eliminates the static error (see Appendix 7.1).

Figure 8.31d is again the same valve, this time with two pressure feedbacks (static load error washout). The stabilization feedback, normally subtractive, is equipped with a high pass filter and therefore preponderates at high frequency, but at low frequencies it is dominated by an additive feedback which eliminates the static error due to the attachment elasticity.

(*4*) *Acceleration-switching valves* (*Figure 8.32*)—Acceleration-switching valves are two-stage valves with no springs or internal feedback, in which an electrical input produces a corresponding acceleration of the main spool. They are supplied with constant-amplitude square-wave input signals modulated in duration (*Figure 8.32b*). The electric motor has only two equilibrium positions, and the commutation frequency is constant and high (200–400 c/s).

This type of servo valve necessitates a fairly powerful electric stage, a special oscillator ahead of it and special compensating circuits, but it can be used with a non-linear motor having hysteresis and a relatively crude valve.

The acceleration-switching servo valve has negligible dead zone and hysteresis, good linearity, good insensitivity to contamination and operates well at high temperature.

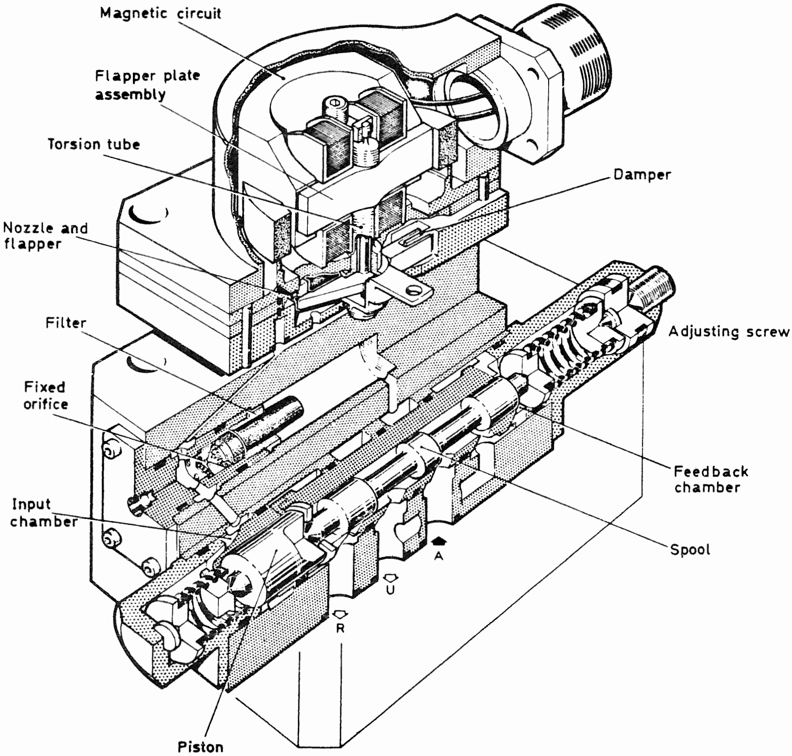


Figure 8.30 (*b*) Messier flow valve A25565

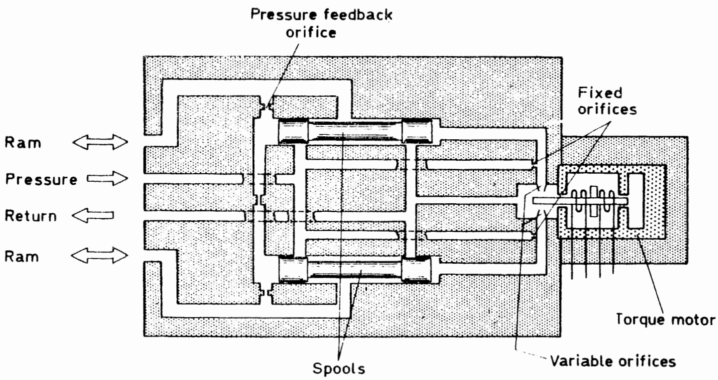


Figure 8.30 (*c*) Cadillac pressure valve

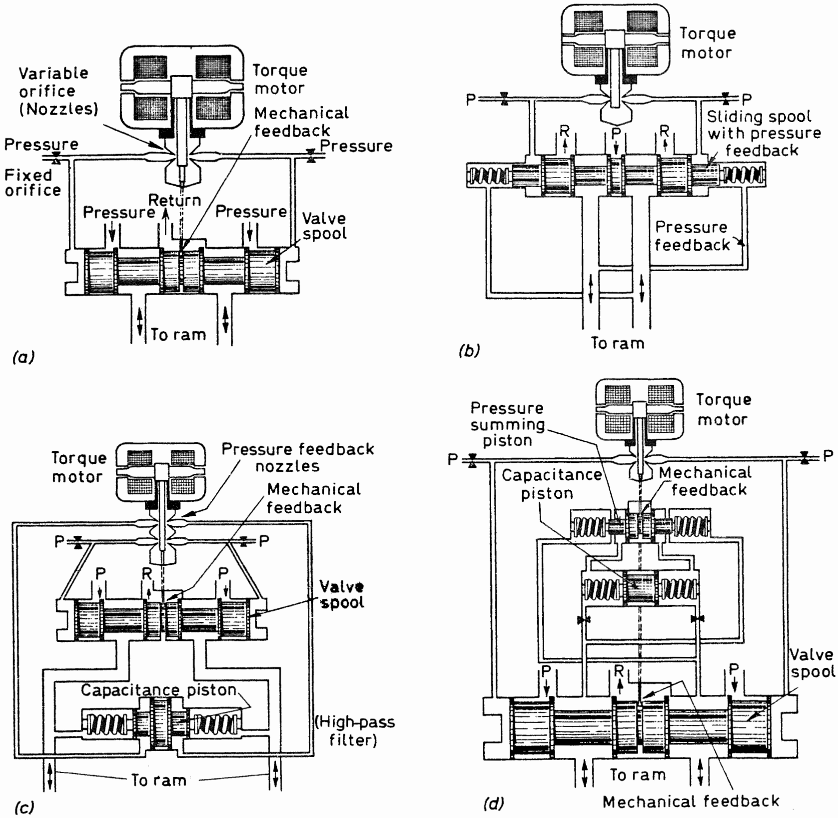


Figure 8.31. Moog pressure-feedback valves

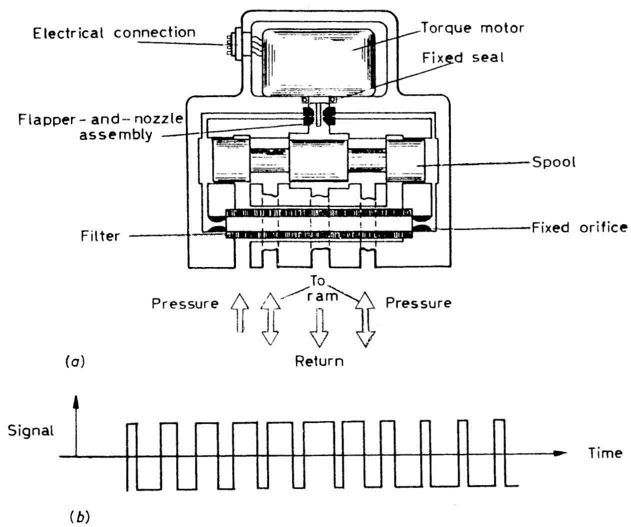


Figure 8.32. Bendix acceleration valve

8.2.4. THE CHOICE OF A SERVO VALVE

8.2.4.1. *Valve or servo valve? Servo valve or electrovalve?*

The servo valve is a component with an electrical input, and its use is therefore justified only in electrohydraulic systems. In other words, the servo valve may replace an ordinary valve only if there are good reasons for introducing electricity into the hydraulic system. An excellent example is the development of aircraft servo controls between, say, the 'Caravelle' and the 'Concorde', as indicated in Section 8.1.

The servo valve gives an output which is proportional to its input as compared with the on–off control normally provided by an electro valve (solenoid valve). The functions are different, and there is therefore no real reason to replace the electrovalve by a servo valve, but many engineers are impressed by the very low input control power of servo valves (of the order of 10^{-1} W compared with 10^1 W for electrovalves); by their high response speed (of the order of 2×10^{-3} sec as against 10^{-2}–10^{-1} for electrovalves); and by their low weight: the latest models weigh less than 0·5 kg (1 lb) for flows up to 900 cm^3/sec or 12 gal/min.

While these advantages may justify the use of a servo valve for on–off control, the following points should be considered before making a final decision. Servo valves are much more expensive than electrovalves. Their high performance is achieved at the cost of a significant permanent flow. Furthermore, the response speed measured for a limited amplitude, of about 10–20 per cent of the maximum, is not conserved for on–off operation, where the valve spool moves from one extreme position to the other. On the other hand, electrovalves with lower input powers and higher response speeds are now becoming available.

8.2.4.2. *Flow or pressure control servo valves?*

(*a*) *Establishing a flow*—In certain cases, particularly in machine tools, we might want to establish a flow of oil, Q, proportional to an electrical input signal, *e*. The simplest solution is to use a *flow control servo valve in an open loop.*

But the great majority of servo valves conform not to the equation $Q = Ke$ but to $S = Ke$ where S is the cross-sectional area of the valve opening. Thus, if

P_1 = supply pressure
F = load
A = effective area of the ram
$P_L = F/A$ = load pressure,

applying Bernoulli's equation, we have

$$Q = K_1 S\sqrt{P_1 - P_L} = KK_1 e\sqrt{P_1 - P_L}$$

i.e. the system will be *sensitive to variations of the supply pressure and of the load.*

These effects are very often small, especially if there is pressure regulation of P_1 and if the ram is made fairly large. To increase the accuracy, we can use (*a*) a true flow-control servo valve, such as the *Pesco* type (*Figure 8.27*); these servo valves, however, are not commercially available;

(*b*) a normal servo valve in a close loop with flow feedback. This is the solution used at present, the flow being controlled by the detection of the speed of the power component;

(*c*) the open-loop configuration can be conserved by the introduction of an elaborate load compensator based on the pressure difference in the two chambers of the ram. This solution could be attractive when the accuracy required is not too high and when there is a difficult stability problem.

(*b*) *Establishing a position*—This is the most usual problem in hydraulic control. It is solved by using a *normal servo valve in closed loop with position feedback.*

Note 1—Since the servo valve is an *integrating* component, it cannot be used in an open loop without null shift.

Note 2—The use of a *true flow control servo valve* to give a dynamic performance independent of the load, although satisfactory in theory, has in practice as many disadvantages as advantages.

Note 3—A servo valve with *secondary pressure feedback* solves the problem of stability equally well, but it is difficult to find a suitable valve on the market.

(*c*) *Establishing a pressure*—This type of problem occurs fairly often (presses, rolling mills, force pilotage of certain missiles, artificial feel for servo controls). It can be solved in two ways:

(*a*) by using a *pressure-control servo valve in an open loop*;

(*b*) by employing a *normal-flow servo valve* in a *closed loop* with *force feedback.*

The first solution is simpler, since a number of suitable pressure-control servo valves are available. Although this would give greater reliability, it has limited accuracy, since the relative accuracy of the force output is, at the most, equal to that of the servo valve (about 1 per cent).

The second solution is more complex and more expensive but gives an accuracy approaching that of the pressure or force pick-up used.

8.2.4.3. *Flow requirements*

The manufacturers' specifications normally give the flow supplied by a servo valve as a function of the electrical input; for one or more values of the nominal supply pressure, P_n; for zero or negligible return pressure; and under no load conditions.

This indicates that the measurements are made with the servo valve attached to a free ram in which both chambers are effectively at the same pressure, very close to $P_n/2$ if the servo valve is symmetrical. In other words, the *no-load* or *nominal flow* is that obtained when *all* the pressure P_n is used to pass the oil through the orifices of the servo valve, $P_n/2$ at the inlet orifice and $P_n/2$ at the outlet orifice if the servo valve is symmetrical.

On the other hand, the engineer designing a hydraulic system needs to obtain a given flow under given conditions. Starting from these conditions, he must be able to calculate the no-load flow in order to select a suitable servo valve for his system.

Let Q be the maximum flow required under a supply pressure P_1, a back pressure P_2 and with a load F applied to a ram of effective area A. The pressure drop causing the flow Q in the valve is $P_1 - P_2 - F/A$.

The elimination of K between the two equations

$$Q = KS\sqrt{P_1 - P_2 - \frac{F}{A}}$$

and

$$Q_n = KS\sqrt{P_n}$$

where S is the cross-sectional area of the valve opening and K the constant $\sqrt{2/\zeta\rho}$, gives an equation for the no-load flow required:

$$\boxed{Q_n = Q\sqrt{\frac{P_n}{P_1 - P_2 - F/A}}}$$

Example—Suppose a ram of effective area 10 cm^2 is to move a load of 900 kg at a speed of 15 cm/sec. The supply pressure available is 100 kg/cm^2 and the back pressure 10 kg/cm^2. A catalogue is available giving the no-load characteristics of a number of valves at nominal pressures of 200 kg/cm^2, and a suitable valve must be selected.

The actual flow under load is

$$Q = 10 \text{ cm}^2 \times 15 \text{ cm/sec} = 150 \text{ cm}^3/\text{sec}$$

The no-load flow is

$$Q_n = Q\sqrt{\frac{200}{150 - 10 - \frac{900}{10}}} = 150\sqrt{\frac{200}{50}}$$

$$\therefore Q_n = 300 \text{ cm}^3/\text{sec}$$

Note—The designer usually has to satisfy a number of operating conditions. The no-load flows for each condition must be calculated and a servo valve selected to satisfy all these conditions. Too large a servo valve must be avoided, since the dead zone, hysteresis and null shift are roughly proportional to the maximum flow, and the use of a valve with too high a maximum flow would have a detrimental effect on the performance of the system.

8.2.4.4. *Special servo valves*

The great majority of servo valves commercially available are *four-way symmetrical* valves like the one described.

Without going into too much detail, we should examine some other types of servo valves which exist.

(*a*) *Asymmetrical flow control servo valves*—The dimensions of the ports

Chamber 1 inlet	($i1$)
Chamber 1 outlet	($o1$)
Chamber 2 inlet	($i2$)
Chamber 2 outlet	($o2$)

may be different; for example, for constant speed using a ram with chambers of different size, $i1 = o1$, $i2 = o2$ but $i1 \neq i2$; likewise, for different speeds in each direction of the ram, $i1 = o2$, $i2 = o1$ but $i1 \neq i2$.

(*b*) *Servo valves with decentred characteristic*—When there is no input current, the main spool takes up a non-central position. This fact allows us to choose the direction of null shift following an electrical failure and is often used for safety reasons when the consequences of a null shift differ according to its direction, as e.g. in the raising or lowering of a hydraulic press.

(*c*) *Three-way servo valves*—These are used for the control of differential-area rams.

8.2.4.5. *Main specification features*

The following is a list of the main features which may be considered by a designer when selecting a servo valve. Their relative importance will obviously vary for different applications.

(*a*) *Geometrical specifications*

Size

Weight

Attachment

Electrical connections

Hydraulic fittings (orifices, layout, dimensions, seals)

Existence of preferred orientation relative to the direction of acceleration.

(*b*) *Electrical specifications*

Number of windings and *layout* (*Figure 8.33*).

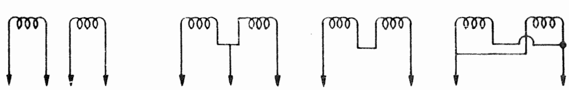


Figure 8.33

Impedance of the windings may, because of the e.m.f. developed by the movable parts of the torque motor, vary with the frequency and amplitude of the input signal (the concept of motional impedance). In some servo valves, the motional impedance is small compared with the impedance measured when the torque motor is locked in position.

Insulation

Need for external dither and, if dither is required, its amplitude and frequency.

Need for quiescent current and its magnitude.

Nominal maximum command current

Maximum current which can be applied without damaging the servo valve (together with an indication of the time for which it can be applied).

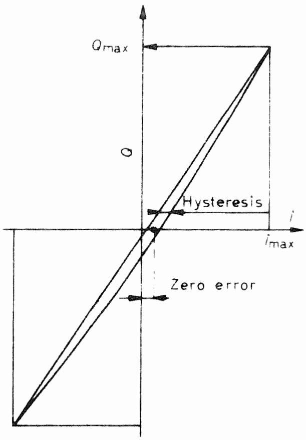


Figure 8.34

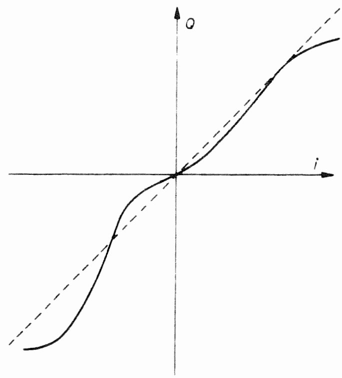


Figure 8.35

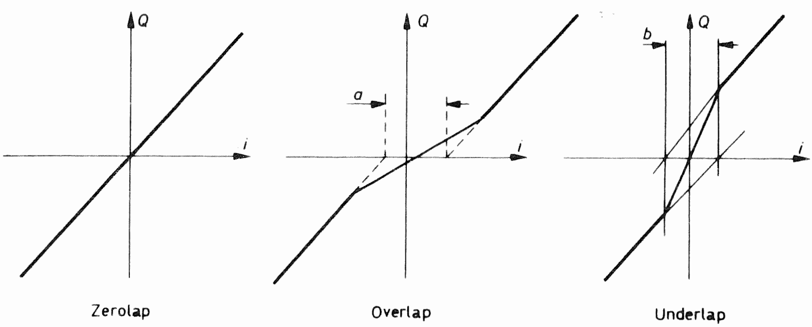


Figure 8.36

(c) *Static performance specifications*

Gain or an indication of the maximum flow corresponding to the maximum input for a given load, supply pressure and fluid (density and viscosity).

Hysteresis, normally expressed in mA (*Figure 8.34*).

Null bias (zero error), in mA (see *Figure 8.34*).

Dead zone (deadband or threshold), defined as the variation of input current necessary to change the direction of the output (i.e. that of displacement of the ram controlled by the servo valve.

Linearity, including lack of symmetry, saturation and any underlap or overlap (*Figure 8.35*).

Zerolap, overlap, underlap (*Figure 8.36*)

Internal leakage, which is very important when evaluating the energy in the system of which the servo valve is a part.

Internal leakage is normally a maximum when the input is zero. It is therefore pessimistic to assume that the leakage flow is constant and equal to its value measured at zero input.

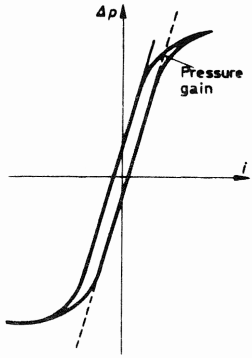


Figure 8.37

Pressure gain (*Figure 8.37*), i.e. the slope of the curve of pressure difference *versus* input current obtained with the ram ports of the servo valve sealed off.

(*d*) *Dynamic performance specifications*

Frequency response curves for an input n per cent of the maximum

Step input response

(*e*) *Mechanical features and effect of ambient conditions*

Maximum supply pressure

Proof pressure

Burst pressure

Maximum return pressure, (*a*) in normal use; (*b*) beyond which there is physical damage to the valve

Filtration

Life (mean time between failures)

Reliability

Maximum and minimum temperatures of the hydraulic fluid

Maximum and minimum ambient temperatures

Maximum accelerations along the three axes

Maximum value of sudden impacts and shocks

Range of vibrations which the servo valve can endure while (*a*) operating, (*b*) not operating

Null shift, variations of gain, dead zone and hysteresis, and any changes of linearity and internal leakage which accompany changes in the parameters quoted, i.e. (*a*) supply and return pressures, (*b*) temperatures, (*c*) accelerations and vibrations.

8.3 THE ELECTRONICS OF ELECTROHYDRAULIC SERVO SYSTEMS*

8.3.1. GENERAL LAYOUT OF AN ELECTROHYDRAULIC SERVO SYSTEM

The normal arrangement of an electrohydraulic servo system is shown in *Figure 8.38*. It illustrates the basic components and signals we shall be considering: the command or input control signal; the feedback and the device providing the feedback signal, the adding component which compares the output signal (feedback) with the input signal and establishes the error signal; the amplifier; the control component (servo valve); and the ram which provides the output.

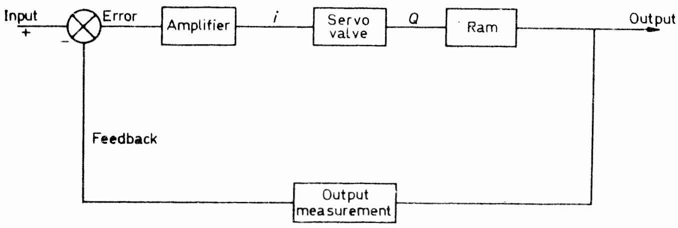


Figure 8.38

8.3.2. THE INPUT (CONTROL SIGNAL)

The input may be *mechanical* in form (displacement, rotation, etc.). It is, in this case, either *compared directly* with the output, if the latter is of the same nature, or *converted into electrical form*.

The input may also be electrical: either a *direct voltage* whose amplitude and sign represent the desired output, or an *alternating voltage* whose amplitude and phase relative to a reference voltage (0 or 180°) represent the desired output. In this case, we say that the input is represented by the *modulation* of a *carrier* frequency.

* This Section has been written by Mr. C. Pascal, head of the Electronics Department of the D.B.A., Air Equipement Division. Taken from a paper presented at a conference on hydraulic control techniques under the auspices of CEGOS, it is a concise account of the problems and solutions peculiar to the electronic components of electrohydraulic servo-mechanisms.

The inputs most commonly encountered correspond to position. They are given by an angular position, by a modulated carrier or, more frequently, by a direct voltage.

8.3.3. FEEDBACK SIGNALS

The output is initially always mechanical: position, force or pressure, velocity. It is converted into an electrical voltage by the use of suitable transducers.

8.3.3.1. *Measuring components*

Position measurement—(*i*) By *potentiometers* (rotary or linear) supplied with either direct or alternating voltage according to the nature of the input. This is the simplest and most direct method, but if wound potentiometers are used, there are several disadvantages:

(*a*) Fragility and wearing out of the sliding contact;

(*b*) Vibrations causing eddy currents of the arm;

(*c*) Bad contacts, destabilizing discontinuities and poor resolution;

(*d*) The difficulty of obtaining potentiometers of standard production.

Recent technical advances, such as developments of metallic film tracks and moulded plastic tracks, have resulted in the development of rotary and linear potentiometers which should not suffer from all the disadvantages mentioned.

(*ii*) By *inductive detectors*, (*a*) with moving coils (synchros, Linvars, etc.); (*b*) with moving iron (Elliott, Sanborn, etc.).

Synchros may be classified as *angular difference detectors*, since neither the input nor the feedback appears in the form of a voltage. However, they have moving contacts (as do Linvars) and are less reliable than machines with fixed contact devices.

Force measurement—(*i*) By strain gauges, but the output signals are very small; (*ii*) by oil-pressure measurement.

Pressure measurement—(*i*) By conversion of pressure to position (piston and spring); (*ii*) by special devices (gauges, potentiometers, inductive devices).

Velocity measurement—(*i*) By displacement of a magnet in a coil; (*ii*) by the differentiation of a position measurement (potentiometers should be avoided in this case, because of their discontinuities).

All these detectors give either a direct or an alternating voltage. The comparison with the input is easier, if this is itself a direct voltage. An alternating voltage must be demodulated if the input is a direct voltage. There are two possibilities:

(*a*) to modulate the input, compare it with the feedback and then amplify the error;

(*b*) to demodulate the feedback, compare it with the input, amplify and then generally modulate again.

The second procedure may seem more clumsy but is nevertheless preferable to the first; the comparison of alternating voltages requires both input and output signals to be of *the same form and in phase with each other*; and if the error signal is a direct voltage, it can be used in *compensating circuits*.

The following two circuits are typical of the arrangements used for the demodulation of feedback signals.

8.3.3.2. *Demodulation circuits*

It will be shown that amplifiers can be controlled either by a *voltage* or a *current*. There is a particular demodulator for each case.

Voltage output (*Figure 8.39*)—Straightforward rectification of the alternating signal does not restore the sign of the measured signal (represented by the phase angle, 0° or 180°, of the alternating signal). Therefore, it is necessary first to add to the signal a synchronous alternating reference voltage of magnitude in excess of the maximum signal voltage. This effects a 'zero shift', and the required rectified signal is obtained, algebraically *added* to a direct mean voltage representing the rectified reference voltage.

By carrying out this operation in a second parallel path but, this time, algebraically *subtracting* the required signal from the reference signal, a second, rectified, signal is obtained. By subtracting this second signal from the first, the initial signal is re-established with its polarity.

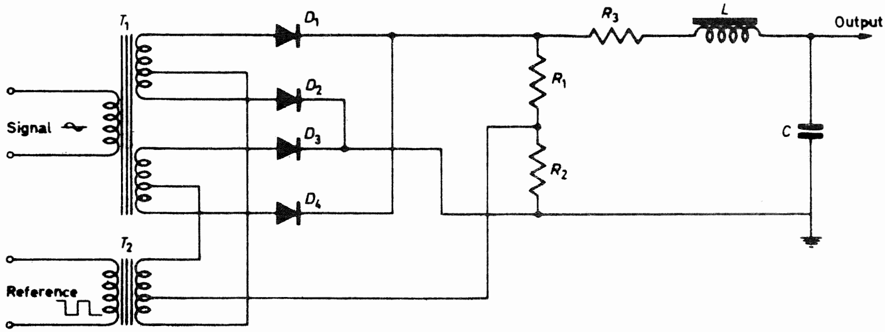


Figure 8.39

In *Figure 8.39*, the transformer T_2 provides the symmetrical reference voltage common to the two full-wave rectifiers (diodes D_1, D_4 and D_2, D_3). In the absence of any signal, this reference voltage causes two equal direct voltages across R_1 and R_2. When a signal is applied (added or subtracted by the secondaries T_1), these two voltages vary in opposite senses, and their filtered difference appears across the terminals of C.

The filter L–R_3–C should obviously have a cut-off frequency significantly in excess of the highest modulation frequencies to be transmitted (see Dynamic Performance, Section 8.3.6), and yet it must eliminate the second harmonic of the carrier frequency*.

Satisfactory performance will thus be obtained only if the *carrier frequency* is *much higher than the maximum transmitted frequency* (e.g. carrier at 2,400 c/s for a modulation up to 100 c/s).

* The residual undulation is passed to the input of the error amplifier; there it is amplified and applied to the servo valve.

Current output (*Figure 8.40*)—The principle is the same, but the second rectification (diodes D_2, D_3) delivers a negative voltage, so that the mean currents due to the reference voltage supplied to the output by R_1 and R_2, cancel each other out.

The absolute value of the two voltages across C_1 and C_2 always varies in the opposite sense, and the effect of these variations is added in the load.

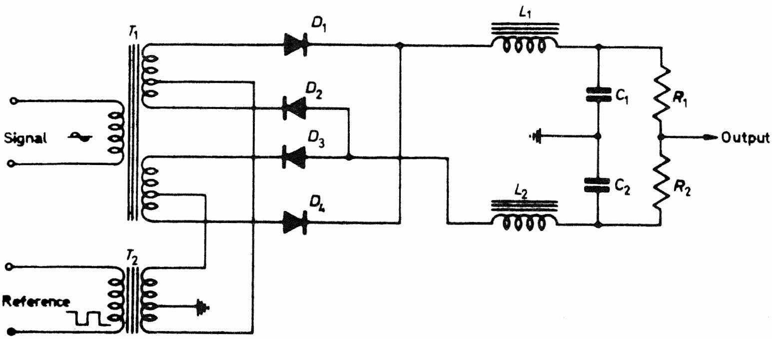


Figure 8.40

Note 1—In *Figure 8.39*, a sinusoidal signal causes a current which is also sinusoidal if the inductance L is sufficiently large and the output current is zero (high load impedance).

In *Figure 8.40*, on the contrary, when an output d.c. flows into the load, the signal source must deliver a rectangular current if the inductance L_1 and L_2 values are high. This current may be detrimental to the performance of this source (for instance, when the detector is adjusted by means of a potentiometric divider).

Note 2—The feedback circuits and the associated demodulators directly affect the *accuracy* of servo mechanisms. Their characteristics are, therefore, very important and should be carefully investigated.

8.3.4. ESTABLISHMENT OF THE ERROR SIGNAL

When the input signal is in mechanical form, it is possible to compare it directly with the output without having to convert both the input and feedback signals into electric voltages.

The simplest and most frequent case is that of angular position, where the difference can be directly and remotely transformed into a voltage difference by means of a chain of synchros.

In other cases, the input and (mechanical) feedback could be placed side by side and a single detector of position (or force, etc.) would give the position (or force, etc.) error signal directly.

Finally, we may prefer to represent a mechanical input by a voltage. If the feedback and the input have the same nature, it is therefore convenient to use *two identical detectors* supplied by the *same source*. But, despite being in a better position because of the identity of the sources of input and feedback, we again face the problem of comparing two voltages, which is different for direct and alternating voltages.

For direct voltages, there is only one real difficulty. If the amplifier input is voltage-driven (high-input impedance), the series arrangement of the three systems, input, feedback and amplification, requires that one be isolated from earth.

For alternating voltages, this isolation is easily effected by means of a transformer, but the *wave form* and *phase angle* must be taken into consideration. The conditions necessary to obtain a zero error signal are:

(*i*) equal voltages (r.m.s. or mean)
(*ii*) absolutely identical wave forms
(*iii*) the two voltages must be exactly (or exactly opposite) in phase.

The last two conditions cannot be achieved with absolute accuracy even with two identical detectors (owing to temperature difference, for example), and can be attained only approximately if different voltage sources are used. (This is particularly true when a direct voltage input is to be modulated before comparison with a *sinusoidal* alternating feedback voltage, since the modulators tend to produce *square* wave forms. The error must therefore be filtered.)

The result is that, even when the mean error signal is zero, the amplifier receives input voltages (residual harmonics or quadrature residual voltages) which can affect its operation and even cause instantaneous *saturation*. This annoying effect results in a loss of sensitivity of the amplifier which therefore causes a reduction in the open-loop gain and a significant deterioration in the performance of the servo system.

8.3.5. AMPLIFICATION OF THE ERROR SIGNAL

8.3.5.1. *Output stage*

The direct or alternating error signal must now be converted into the command current for the servo valve. A servo valve has two coils with current I_0 passing through them in opposite directions when there is no command signal. When a command current ($\Delta I = I_1 - I_2$) is present, the currents in the coils become

$$I_1 = I_0 + \Delta I/2$$
$$I_2 = I_0 - \Delta I/2$$

The servo valve should be controlled by a class A push-pull stage (*Figure 8.41*).

In theory, the value of I_0 is of no importance, since the amp turns $N_1 I_0$ and $N_2 I_0$ cancel each other out. In practice, the zero of the valve varies slightly with I_0, and it is advisable to keep I_0 constant to within, say, 20 per cent.

The hydraulic flow through the valve depends on the command current ΔI. Now, the resistances of the two coils will not be exactly equal and, in any case, will vary with temperature, increasing by 40 per cent when the coils are immersed in oil at 120°C. Also, the inductance of the coils is not negligible (of the order of 10 H for coils of 1,000 Ω resistance, so that $L\omega \triangleq 3{,}000\ \Omega$ at 50 c/s). The result is that the *voltage* across the terminals of the servo valve is never a meaningful parameter, and in view of both the static performance (gain) and the dynamic performance (phase displacement) it is necessary that the amplifier should have a *high internal output resistance*, so that the command current which it supplies does *not depend on the impedance of the servo valve.*

Because of the high inductance of the coils, any small residual undulation (carrier frequency) of the current in the valve causes a *high undulating voltage* across the valve terminals. This affects the amplifier only, since the current is only slightly modulated, and in any case the valve has poor response at the very high frequencies used as carrier. It is even possible to apply a slight current undulation deliberately in order to reduce the dead zone of a valve (dither).

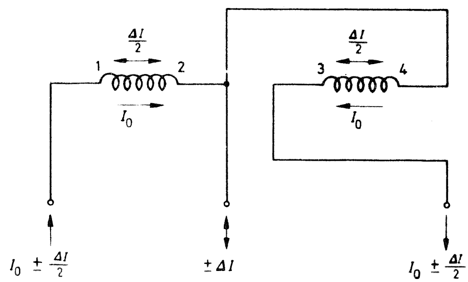


Figure 8.41

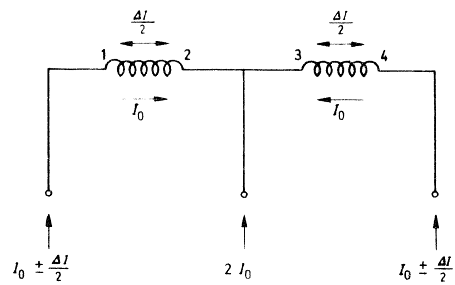


Figure 8.42

The maximum flow demanded from a servo valve may be less than that which it is able to supply (hydraulic saturation). It is therefore advisable to *limit the command current* to a suitable value (electric saturation).

Summing up, then, the output requirements for a suitable amplifier are

(*i*) class A push-pull;
(*ii*) high internal resistance;
(*iii*) filtered or protected against high output voltages;
(*iv*) well defined maximum current.

Plate 3 shows a typical servo amplifier on a printed circuit.

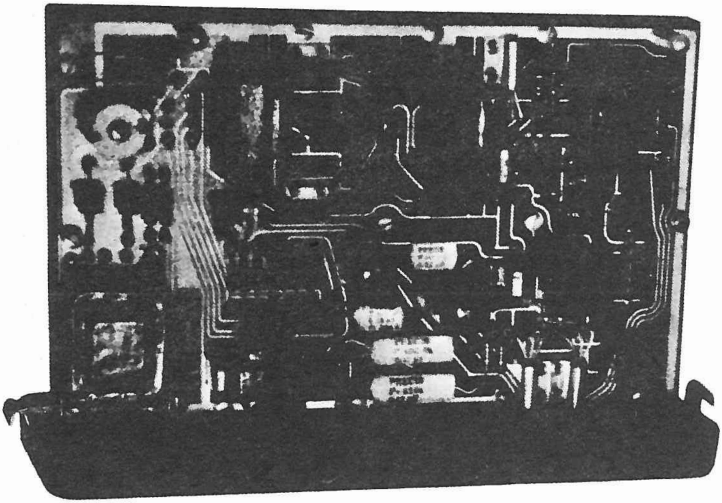


Plate 3. Amplifier

8.3.5.2. *Examples of output stages*

The two coils of the torque motor may be connected in one of two different ways:

(*i*) *Parallel* push–pull (*Figure 8.41*)
(*ii*) *Series* push–pull (*Figure 8.42*).

Figures 8.43 and *8.44* show the layout of two output stages (parallel push and series push) controlled by the demodulators described already (a.c. carrier pre-amplification). Note that

(*i*) The circuits are *symmetrical* and the zero drift should therefore be small.

(*ii*) The negative feedback of current by the emitter resistances ensures high internal resistance for I_0 and ΔI.

(*iii*) I_0 is produced by the demodulator reference voltage (carrier) and the command current by the signal (modulation). Therefore, the ratio chosen between I_0 and ΔI_{max} defines that of the reference voltage and the maximum signal amplitude.

(*iv*) The reference and signal voltages are rectangular; any undulation on the valve current will therefore be small.

(*v*) The collectors of the transistors are protected against high output voltages either by a filtering condenser (tuning the coils to a frequency greater than those of modulation and less than that of the carrier) or by a reversely mounted diode.

(*vi*) The supply voltage (or voltages) should be compatible with the maximum resistance of the coils when hot.

(*vii*) In the series layout (*Figures 8.42* and *8.45*) the command current can be isolated in a resistance, measured and used to establish a feedback system; this improves the zero, makes the gain more constant and increases the internal resistance and the band pass.

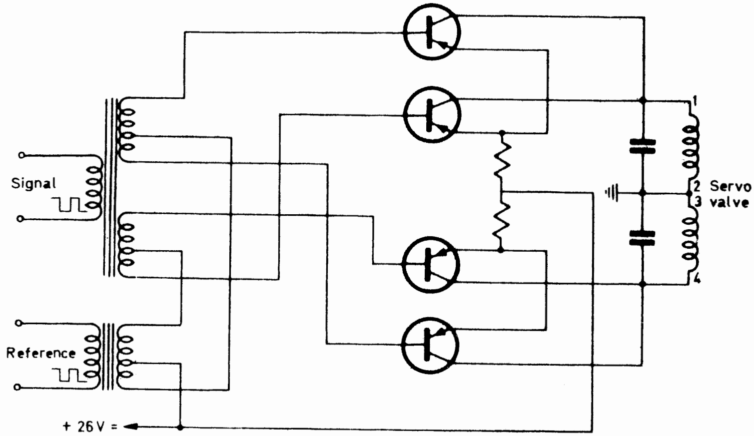


Figure 8.43

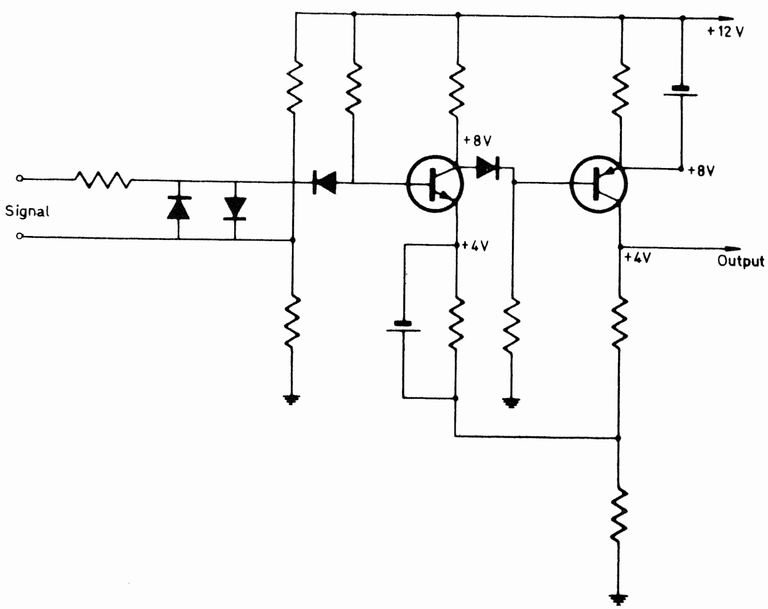


Figure 8.44

8.3.5.3. *Limitation*

For a push–pull stage, limitations should be applied to the base driving voltages (the resistance of the coils is variable). But it is simpler, and equivalent, to limit the error signal supplied by the preamplifier, either by Zener diodes or more simply, if the supply voltage is stable, by natural saturation of the last stage.

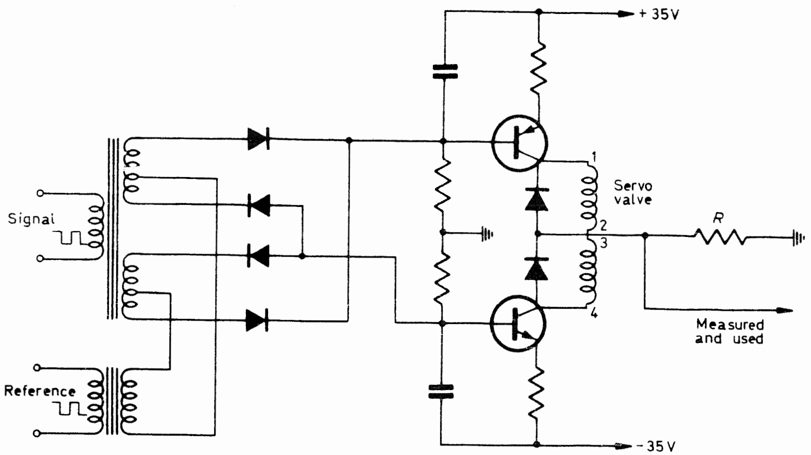


Figure 8.45

8.3.5.4. *Preamplification*

The only special requirement necessary in a servomechanism preamplifier, besides the limitation of output signal mentioned above, is a capacity to accept input signals much larger than those necessary for the production of output signals.

The maximum speed of the ram is, in fact, always obtained by a position error signal which is only a fraction of the maximum position signal. It is useful to establish the ratio of the maximum input signal to the input (position error) necessary to obtain the maximum speed (and, therefore, the maximum command current of the servo valve). This dimensionless ratio is known as the *reduced gain*, G_r, and it may be used for comparing different servo systems independently of the values of the components used in the specific systems considered.

A preamplifier may, therefore, especially when being switched on, receive up to $2G_r$ times the normal input voltage. (G_r may easily be 10 or more).

This input overload can result in either

(*i*) eventual damage to the amplifier (for example, by too high an inverse base–emitter voltage) or

(*ii*) bias shift, retained by the by-pass and coupling condensers.

These bias shifts can be very serious: as they do not disappear immediately the cause of saturation is removed (condensers), and after some time the amplifier may either fail to operate or may give aberrant output voltages.

The following precautions may be used to overcome these disadvantages:

(*i*) limiting the input signals by diodes connected in parallel (several may be placed in series ending with Zener diodes if the nominal input level is high);

(*ii*) limiting the transistor signals by diodes in series;

(*iii*) choosing the transistors' operating point to be in the middle of the load characteristic (letting the polarization current be half the maximum current for saturation). *Figure 8.44* illustrates these precautions.

8.3.5.5. *Input characteristics*

The input to the amplifier may be either a *voltage* or a *current*.

Voltage input—There is obviously an input current but it is very small because the input *impedance* is *very high* in comparison with the internal resistance of the source. Hence, the gain is defined in mA/V.

Such a characteristic may be obtained by a *series* feedback, applied for instance to the first stage. For the comparison of input and feedback with this type of input, one of the sources, or the amplifier, must be *isolated*. We may add a number of signals only if they are isolated.

Current input—There is obviously an input voltage, but it is very small, since the input impedance is small compared with the total resistance allowing currents proportional to the input and feedback to be applied to the amplifier. Hence, the gain is defined in mA/mA.

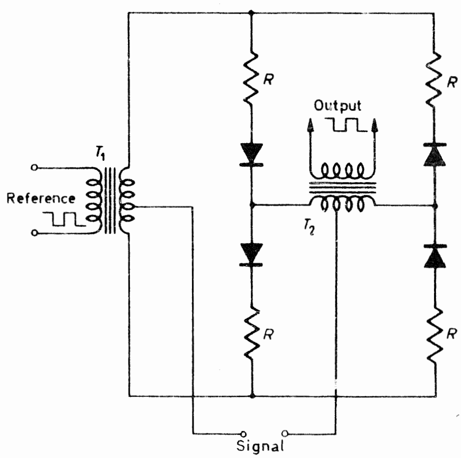
This type of characteristic is obtained by *parallel* feedback to the input.

Additional inputs may be accepted by adding summation resistances. This type of amplifier is given the name 'operational'; it is the type used in analogue computers. The series push–pull described is well suited to this type of input.

8.3.5.6. *Modulators*

It is possible to amplify a direct voltage directly, but the precautionary measures necessary make it preferable in certain cases to amplify the signal in alternating form, on a carrier. The direct error signal must therefore be modulated, and either of the following circuits may be used for this purpose.

Figure 8.46



Ring modulator (*Figure 8.46*)—Two dividers using equal resistances R are supplied with a symmetrical rectangular reference voltage. Each of the dividers conducts during one half cycle, so that the direct input voltage is successively applied to the two halves of the primary of T_2 and is therefore transformed into a rectangular voltage.

The false zero of this modulator is some tens of millivolts. Its parallel impedance depends on the inductance of the transformer which can be very high. Its series impedance is equal to $R/2$. The maximum input voltage depends only on the reference voltage applied to the bridge ($\frac{1}{4}$ of the total applied voltage).

This modulator has a linearity better than 1 per cent and is suitable for the modulation of signals of several volts and for amplifiers with high input resistance. It is insulated from earth.

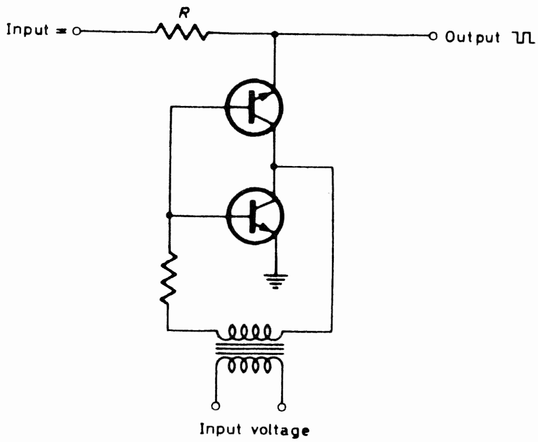


Figure 8.47

Transistorized chopper (*Figure 8.47*)—The signal is simply earthed during $\frac{1}{2}$ of the cycle by means of one or, preferably, two transistors the base of which is, or is not, driven. The use of two transistors mounted in series head to foot compensates the e.m.f. which appears between the collector and emitter in the presence of the base current. Inverse connection (collector and emitter interchanged) gives the best results.

The false zero of this modulator is several mV which may be reduced to 1 mV by sorting transistors. Its parallel impedance is very high: it depends on the cut-off current of the transistors and does not obey Ohm's law. Its series impedance is zero if the resistance R is provided by the external circuit. Its gain is 0·5.

This modulator is simpler than the previous one and mainly used at the input of operational amplifiers because of its small virtual zero.

8.3.5.7. *Amplifier static performance accuracy*

The importance of the performance of feedback circuits has already been discussed. The accuracy obtained (linearity, stability, artificial zero, dead zone, etc.) directly affects that of the servomechanism: these circuits are, in fact, its measuring components.

This also applies to comparison circuits, but their construction is normally such that there is no serious problem of accuracy.

Fortunately, the static performance requirements for error amplifiers are much less stringent and, in general, for all components dealing with the error signal. In fact,

(*i*) A variation of the static or differential gain (linearity) affects only the response speed and the stability margin which seldom need to be kept absolutely constant.

(*ii*) A false zero (or a zero drift) is partially compensated by the feedback loop.

It is convenient to relate the zero error to the input of the amplifier (or comparison circuit), i.e. the value of the error signal (expressed as percentage of the maximum control signal) which gives zero amplifier output current or, better, zero servo valve flow. This procedure gives the relative error due to the false zero; it can also be obtained by dividing the false output zero, expressed as percentage of the maximum output, by the reduced gain.

8.3.6. DYNAMIC PERFORMANCE

In order to obtain the best possible performance from a servomechanism, the gain of the loop (and, finally, of the amplifier) is increased to the highest value compatible with the stability of the system. This gain will be increased if the frequencies at which the phase displacement reaches 180° in open loop are increased. The problem of increasing the gain, therefore, more or less becomes that of decreasing the phase displacement.

It is much easier, or much less expensive, to increase the natural or cut-off frequencies of an electronic system than those of a hydraulic or mechanical system. Therefore, for the range of frequencies considered in the analysis of a hydraulic system (in practice, up to the compressibility frequency), the transfer function of the associated electronic system will be kept constant.

8.3.7. COMPENSATING CIRCUITS

This reasoning leads to the possibility of adjusting the form of the electronic transfer function so as to improve that of the total system. The resulting improvements may be an increase either in the stability margin or in the gain of the loop, giving rise to a higher response speed.

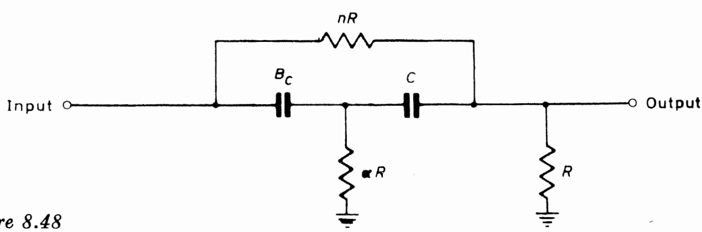


Figure 8.48

A particularly attractive form of compensation is obtained from the network shown in *Figure 8.48* (bridged-*T* network). Its transfer function is

$$\left(\frac{1}{n+1}\right)\left[\frac{1+2\xi_1\dfrac{p}{\omega_1}+\left(\dfrac{p}{\omega_1}\right)^2}{1+2\xi_2\dfrac{p}{\omega_2}+\left(\dfrac{p}{\omega_2}\right)^2}\right]$$

where

$$\omega_1 = \frac{1}{RC\sqrt{n\alpha\beta}}$$

$$\frac{\omega_2}{\omega_1} = \sqrt{n+1}$$

$$\xi_1 = \frac{1}{2\sqrt{n}}(1+\beta)\sqrt{\frac{\alpha}{\beta}}$$

$$\frac{\xi_2}{\xi_1} = \frac{\omega_2}{\omega_1}+\left(\frac{n}{\sqrt{n+1}}\right)\left[\frac{1}{\alpha(1+\beta)}\right]$$

The transfer function of a loaded ram is

$$\frac{\omega_f}{p\left[1+2\xi\dfrac{p}{\omega_c}+\left(\dfrac{p}{\omega_c}\right)^2\right]}$$

If we make $\xi_1 = \xi$ and $\omega_1 = \omega_c$, the overall transfer function for the network and ram becomes

$$\left(\frac{\omega_f}{n+1}\right)\left\{\frac{1}{p\left[1+2\xi_2\dfrac{p}{\omega_2}+\left(\dfrac{p}{\omega_2}\right)^2\right]}\right\}$$

The polynominal representing the ram is replaced by another polynominal whose coefficients (natural frequency and damping) are larger and can to a certain extent be chosen.

The problem has therefore been replaced by a simpler one. In practice, we are obviously limited, first by the characteristic equations of the compensating network in the choice of new polynominal and then, in particular, by the variability of the characteristics of the loaded ram, dependent on variable or indefinite parameters (oil temperature, pressure, ram position, friction, etc.). The compensation of the resonance of the ram, which is more necessary if the damping is small, will be less effective if this resonance is more peaked.

The improvement which may be expected from the use of this type of compensating network is an increase of a few decibels in the open-loop gain: to double the gain would be an extremely good result.

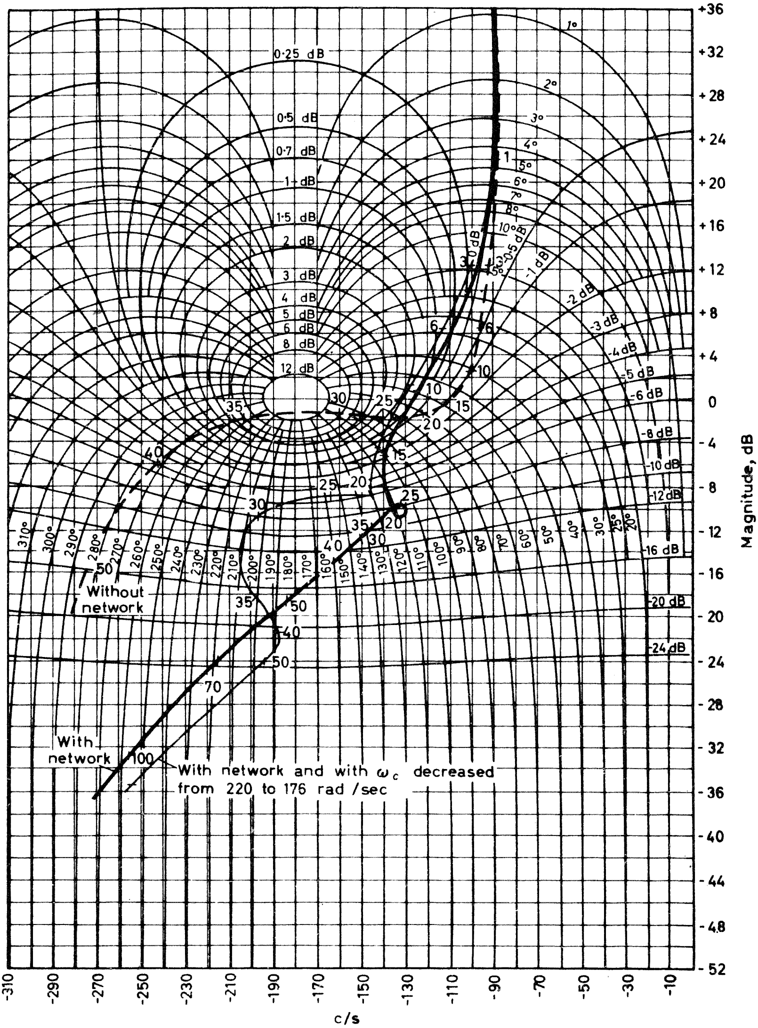


Figure 8.49. Effect of compensating network

There are obviously many other types of compensating networks. They vary according to the form of the servo system and the characteristics required.

Electrical circuits are well suited to many configurations. Many elementary networks may be added to an electronic servo system: to the feedback circuit; to the error signal; to the amplifier feedback, etc. Note, in passing, that these networks are applicable only to unmodulated signals, the frequency accuracy required preventing the band-pass filters from acting on the side bands of a modulated carrier.

Figure 8.49 shows Nichols diagram for a position-feedback system. The curves illustrate the effects of adding a compensating network and of varying the compressibility frequency.

This Section has shown that the effects of the physical components on the performance of an open loop may be diminished but not eliminated. The use of a compensating network may result in a limited improvement in the overall system but will not change it to such an extent that it becomes unrecognizable.

REFERENCE

[1] Kinney, W. and Weiss, P. *Appl. Hydraul. Pneumat.* 12 (1959/2) 68

9

THE PERFORMANCE OF HYDRAULIC SERVO SYSTEMS

9.1. INTRODUCTION

In the initial planning stage in the design of a servo system, when a hydraulic system is being considered and compared with other methods of control, it is necessary to make a quick estimate of the maximum performance which can be expected. Sometimes also, before a system is actually constructed, the quality of its design must be assessed in relation to the best performance possible using modern techniques. This is the problem of estimating the maximum performance of hydraulic servo systems.

The two preceding Chapters, dealing with hydraulic and electrohydraulic servo systems, contain all the material necessary for this estimation. In this Chapter, the information is merely rearranged and developed so as to be more readily usable by the reader.

The distinction between hydraulic and electrohydraulic servo systems, essential in a descriptive account of these systems, is not maintained here since it is irrelevant in determining the maximum performance.

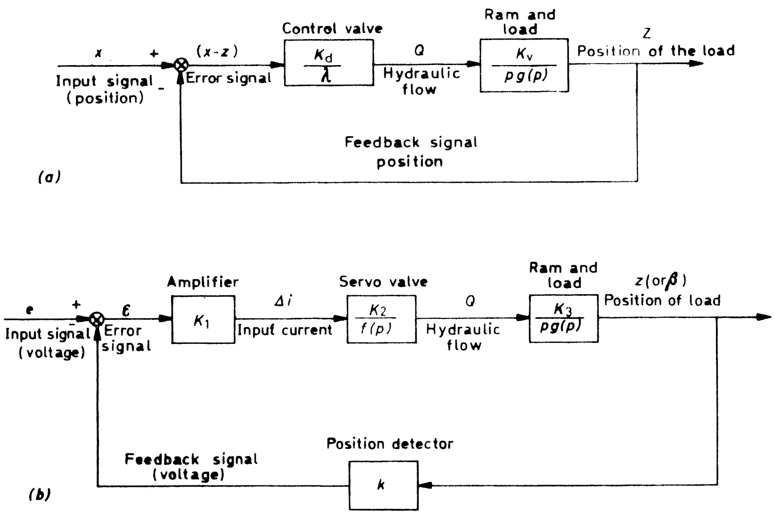


Figure 9.1. (*a*) Hydraulic servo system; (*b*) electrohydraulic servo system

9.2. SPEED OF RESPONSE OF POSITION FEEDBACK SYSTEMS

Figures 9.1a and *b* give the block diagrams of the classical hydraulic and electro-hydraulic servo systems with position feedback analysed in Chapters 7 and 8.

For the *hydraulic system*, the transfer function of the ram is obtained by eliminating the intermediary pressure variable between the flow equation and the force equation. For an equal-area double-acting ram, with the assumptions and notation of Section 7.3, these are

$$Q = S\frac{\mathrm{d}z}{\mathrm{d}t}+\frac{V_t}{2B}\frac{\mathrm{d}(P_A-P_B)}{\mathrm{d}t}$$

$$(P_A-P_B)S = rz+f\frac{\mathrm{d}z}{\mathrm{d}t}+m\frac{\mathrm{d}^2z}{\mathrm{d}t^2}$$

The result is

$$\frac{z}{Q} = \frac{1}{Sp\left(1+\dfrac{r}{r_h}+\dfrac{f}{r_h}p+\dfrac{m}{r_h}p^2\right)}$$

where

$$r_h = 2BS^2/V_t,$$

the hydraulic stiffness of the ram full of oil.

Introducing the transfer function of the valve and its control lever which is constant with the value K_v/λ (see Section 7.2), neglecting r in comparison with r_h and substituting

the open loop gain

$$\frac{K_v}{\lambda S} = \omega_f$$

ω_c, the natural angular frequency of the ram + load (valid for those attachment and linkage stiffnesses high enough not to affect ω_c)

ζ, the reduced damping coefficient whose value is always very small, the open-loop transfer function becomes

$$\boxed{(TF)_0 = \frac{\omega_f}{p\left(1+2\zeta\dfrac{p}{\omega_c}+\dfrac{p^2}{\omega_c^2}\right)}} \qquad (1)$$

For the electrohydraulic system, the transfer function of the ram is the same. The transfer function of the whole open loop is therefore

$$(TF)_0 = \frac{kK_1K_2}{f(p)}\left[\frac{1}{Sp\left(1+2\zeta\dfrac{p}{\omega_c}+\dfrac{p^2}{\omega_c^2}\right)}\right]$$

where

k = gain of position detector (voltage/displacement)
K_1 = amplifier gain (current/voltage)
$K_2/f(p)$ = transfer function of servo valve
K_2 = its static gain (flow of oil/current)

One should note that this equation is based on the assumption that the transfer functions of the amplifier and position detector are effectively constants in the range of frequencies considered for the position feedback system, i.e. angular frequencies significantly greater than ω_c. This assumption is usually justifiable, although the author has encountered some unfortunate exceptions with amplifiers and detectors supplied by 50 c/s and even 400 c/s.

If ω_f is the new open-loop gain (this time, kK_1K_2/S), and since it is possible to obtain servo valves whose phase lag and amplitude reduction are small at ω_c, the open-loop transfer function of the electrohydraulic servo system is

$$\boxed{(TF)_0 \sim \frac{\omega_f}{p\left(1+2\zeta\dfrac{p}{\omega_c}+\dfrac{p^2}{\omega_c{}^2}\right)}} \qquad (1')$$

i.e. exactly the same form as the open-loop transfer function of the purely hydraulic servo system.

We know (cf. *Figure 7.6*) that for transfer functions of this form with low values of ζ, the critical frequency is ω_c (phase displacement of 180° and resonance peak). Therefore, a good stability criterion is a gain margin of at least 6 dB, i.e. a $(TF)_0$ magnitude equal to or less than $\frac{1}{2}$ for $\omega = \omega_c$.

Now, for $\omega = \omega_c$, since $1+p^2/\omega_c^2 = 0$, $(TF)_0$ reduces to

$$\frac{\omega_f}{2\zeta\dfrac{p^2}{\omega_c}} = \frac{\omega_f}{2j^2\zeta\omega_c}$$

The criterion therefore becomes

$$\boxed{\omega_f \leqslant \zeta\omega_c} \qquad (2)$$

Practical experience indicates that it is difficult, without great complication, to obtain values of $\zeta > 0{\cdot}25$, so that a reasonable maximum value for the gain is

$$\boxed{\omega_f = \frac{\omega_c}{4}} \tag{3}$$

If we now suppose that the frequency of the oscillations is limited to a value significantly less than ω_c, we may consider that, as far as performance is concerned, but obviously not stability, the system is equivalent to a first-order system having an open-loop transfer function ω_f/p and, therefore, a closed-loop transfer function of $1/(1+p/\omega_f)$.

This conclusion follows:

The highest response speed which may be achieved by a simple hydraulic or electrohydraulic servo system of natural angular frequency ω_c *is that of a first-order system with a time constant* $\tau_{\min} = 4/\omega_c$.

Let us quickly re-examine some of the results for ω_c. Its general expression is

$$\boxed{\omega_c = \sqrt{\frac{1}{m\left(\dfrac{1}{r_h}+\dfrac{1}{F}+\dfrac{1}{N}\right)}}} \tag{4}$$

where

m = movable mass, i.e. in practice, the mass of the load applied to the ram

F = stiffness of the ram mounting

N = stiffness of the linkage between ram and load

r_h = hydraulic stiffness of the ram filled with oil with the valve closed

The expression for the hydraulic stiffness, r_h, is

$$\boxed{r_h = \frac{2BS^2}{V_t} = \frac{2B}{P_1}\frac{k_s}{k_v}\frac{F_R}{z_M}} \tag{5}$$

where

B = bulk modulus of oil

S = effective area of ram

V_t = half of the total swept volume of the ram (see Section 7.3.1)

P_1 = supply pressure

k_s = power coefficient (see Section 7.3.1)

k_v = volumetric coefficient (see Section 7.3.1)

F_R = maximum external resisting force

z_M = half the maximum stroke

When the output is rotational, the expression for ω_c is simply modified by replacing the mass m by the movement of inertia, I, and the stiffnesses (force/unit length) by the angular stiffnesses (torque/rad), while the expression for r_h becomes

$$\boxed{r_h = \frac{2BS^2l^2}{V_t} = \frac{2B}{P_1}\frac{k_s}{k_v}\frac{C_R}{\beta_M}} \tag{5'}$$

where

C_R = maximum resisting torque

β_M = half the maximum angular stroke

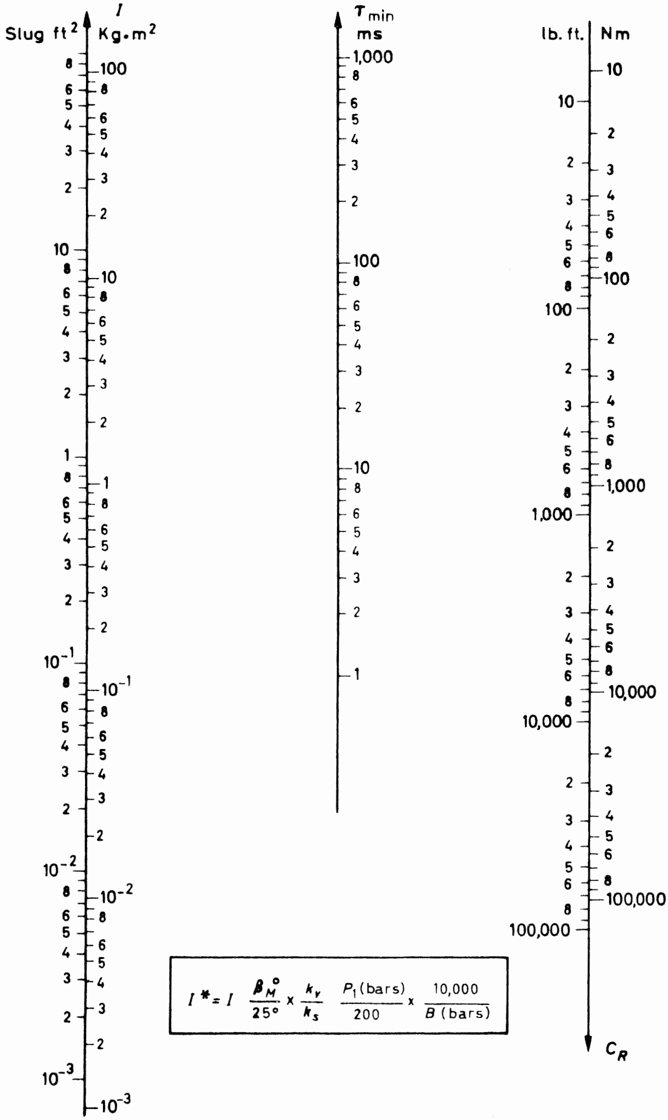


Figure 9.2

The nomograph of *Figure 9.2* gives $\tau_{\min}$ directly as a function of I and C_R for the following case:

F and N large* compared with r_h (which should be true for all well designed hydraulic systems, cf. Section 7.8.1.2)

$B = 10{,}000$ bars
$P_1 = 200$ bars
$\beta_M = 25°$
$k_s/k_v = 1$

For different values of B, P_1, β_M, k_s and k_v the nomograph may still be used if I is replaced by

$$I^* = I\frac{\beta_M^\circ}{25^\circ}\frac{k_v}{k_s}\frac{P_1\text{ (bars)}}{200}\frac{10{,}000}{B\text{ (bars)}}$$

Note—This whole Section deals with servo systems in which the hydraulic motor is a linear actuator. If there is a rotary actuator, the results are not so simple, and the reader is referred to Section 7.9.3.

9.3. ACCURACY OF POSITION-FEEDBACK SYSTEMS

Since the hydraulic servo system with position feedback effects an integration in its final component, the ram, *linear theory* predicts that an 'upstream' component cannot cause a static error in the output. In practice, such errors do exist, owing to certain *non-linearities* in these other components.

Consider the case of a *purely hydraulic* servo system with a dead zone $\pm\epsilon$ in the valve. Suppose that initially the system is in equilibrium with the valve in the middle of its dead zone when, at constant input, a perturbation causes a shift of the output z. This shift causes a valve opening $\Delta y = \Delta z/\lambda$. When Δy reaches ϵ, and only then, the servo system reacts to correct the error.

The maximum shift of the output, known as the output error, is therefore

$$\boxed{\Delta z_M = \lambda\epsilon} \qquad (6)$$

In the case of the *electrohydraulic* system, the calculation of the error due to a valve dead zone is made in the same way, letting Δi_s be the absolute dead zone

* This condition may be expressed as

$$F\text{ (or }N) \gg \frac{2B}{P_1}\frac{k_s}{k_v}\frac{C_r}{\beta_m}$$

i.e. in the general case, where k_s/k_v is in the neighbourhood of 1, B is of the order of 10,000 bars and P_1 of the order of 200 bars,

$$F\text{ (or }N) \gg 100\,\frac{C_r}{\beta_m}$$

The attachment (or linkage) stiffness should be considerably greater than 100 times the stiffness of a torsion spring which would give a resisting torque C_R when deflected through an angle β_M.

of the servo valve, i.e. the smallest current required to operate it, and $s = \Delta i_s/\Delta i_M$ the relative dead zone, i.e. the ratio of the absolute dead zone to the maximum input current.

The same reasoning as above shows that the maximum output error is given by

$$\Delta\beta_M \, k \, K_1 = \Delta i_s$$

Multiplying this equation, term by term, by that defining Δi_M

$$\Delta i_M K_2 K_3 = \left(\frac{\mathrm{d}\beta}{\mathrm{d}t}\right)_M$$

and putting $kK_1K_2K_3 = \omega_f$, we get

$$\boxed{\Delta\beta_M = \frac{s}{\omega_f}\left(\frac{\mathrm{d}\beta}{\mathrm{d}t}\right)_M} \qquad (7)$$

The output error is proportional to the dead zone of the servo valve (of course); inversely proportional to the gain (in keeping with servo system theory), and proportional to the maximum output speed, which simply means that a servo valve for large flows is less accurate in the neighbourhood of the zero.

If we now consider the reduced gain, G_r, of the chain, defined in Section 8.3.5.4 as the ratio of the maximum input signal, e_M, to the error signal, ϵ_i, necessary and sufficient to give the maximum output speed, we have

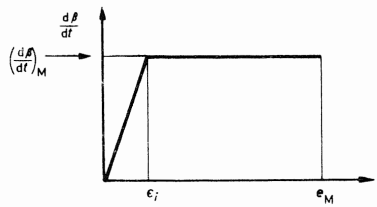


Figure 9.3

$$\epsilon_i K_1 K_2 K_3 = \left(\frac{\mathrm{d}\beta}{\mathrm{d}t}\right)_M \quad \text{(definition of } \epsilon_i\text{)}$$

$$\frac{e_M}{e_i} = G_r \quad \text{(definition of } G_r\text{)}$$

$$e_M = k\beta_M \quad \text{(relation between input and output)}$$

wherefrom, by eliminating ϵ_i and e_M

$$\boxed{G_r = \omega_f \frac{\beta_M}{(\mathrm{d}\beta/\mathrm{d}t)_M}} \qquad (8)$$

and by substituting this relation for G_r in eqn. (7)

$$\boxed{\frac{\Delta\beta_M}{\beta_M} = \frac{s}{G_r}} \tag{9}$$

giving the important result:

The relative output error is equal to the relative dead zone of the servo valve divided by the reduced gain.

Table 9.1

		Model A	*Model B*
Maximum rotational amplitude	β_M(rad)	0·6	0·1
Maximum rotational speed	$\left(\frac{d\beta}{dt}\right)_M$ (rad/sec)	4	0·5
Moment of inertia of load	I (kg.m^2)	0·3	130
Maximum opposing torque	C_R ($m.N$)	200	5,000
Supply pressure	P_1 (bars)	200 (2,850 lb/in.^{2})	200
Bulk modulus of oil at operating temperature	B (bars)	10,000 (140,000 lb/in.^{2})	10,000
k_v/k_s	dimensionless	0·6	0·8
Estimated minimum time constant	$\tau_{\min}$ (msec)	9·4	18
Actual open-loop gain	ω_f (1/sec)	100	45
Corresponding time constant	τ (msec)	10	22
Reduced gain	G_r (dimensionless)	15	9
Zero error of servo valve (dead zone + hysteresis + thermal drift)	s (dimensionless)	4 per cent	4 per cent
Relative accuracy of the chain	$\frac{\Delta\beta_M}{\beta_M}$ (dimensionless)	0·25 per cent	0·5 per cent
Maximum power, $H_M = C_R\left(\frac{d\beta}{dt}\right)_M$	H_M (W)	800	2,500

Numerical examples—Table 9.1 gives the numerical values for two recently constructed hydraulic servo systems with position feedback which did not require electronic or hydraulic compensating networks.

9.4. FORCE FEEDBACK SYSTEMS

9.4.1. DEFINITION

In the case of the *position feedback* system, the output component takes up a *position proportional to the input signal* and as independent as possible of the external forces applied to it.

In the *force feedback system*, the force opposing the actuating component is compared with the input signal. In other words, the output component tends to take up, at each instant, *the position in which it meets an opposing force equal to that demanded by the input.*

There are many applications of the force feedback system: in metallurgy, for presses and rolling mills; in the laboratory, for fatigue-testing machines in automotive engineering, for power brakes, etc.

This type of servo system, sometimes called a *pressure* feedback system since the quantity detected is not really the opposing force but the pressure developed in the actuating component, may take any one of the three forms:

the force servocontrol (Section 7.9.5);

the electrohydraulic servo system (Section 8.2.4.2), based either on (*a*) a servo valve with pressure feedback or (*b*) a normal flow servo valve in conjunction with a pressure detector.

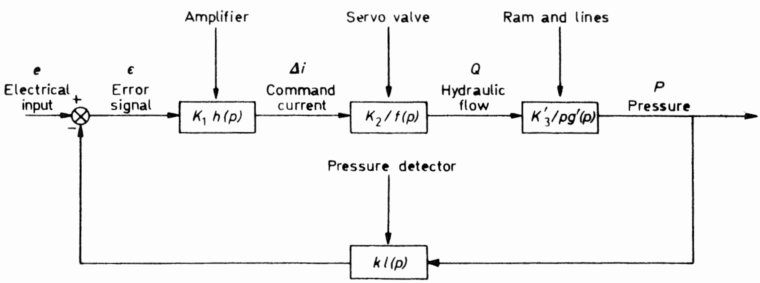


Figure 9.4

Force feedback systems have a different function from position feedback systems, and it will be shown that they also behave very differently.

For this purpose consider the electrohydraulic force feedback system with a flow servo valve and pressure detector (*Figure 9.4*).

9.4.2. PRELIMINARY CONSIDERATIONS OF THE TRANSFER FUNCTION

For the *position* feedback system, the transfer function of the ram is z/Q; it is obtained by eliminating the pressure variable P between the flow and force equations, i.e. for a differential-area ram between

$$Q = S\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{V_t}{B}\frac{\mathrm{d}P}{\mathrm{d}t}$$

and

$$PS = rz + f\frac{\mathrm{d}z}{\mathrm{d}t} + m\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}$$

The result is

$$\frac{z}{Q} = \frac{K_3}{pg(p)} = \frac{r_h}{Sp(r + r_h + fp + mp^2)}$$

where r_h is the hydraulic stiffness defined in Section 7.5.1.

It has been shown that the form of this third-order transfer function, with its integration and poorly-damped trinomial, is responsible for all the deficiencies of position feedback systems.

For the *force* feedback system, with which we are now concerned, the transfer function, P/Q, is obtained by eliminating the position variable z between the *same flow and force equations*. Using the same notation, the result is

$$\boxed{\frac{P}{Q} = \frac{K'_3}{pg'(p)} = \frac{B(r + fp + mp^2)}{V_t p(r + r_h + fp + mp^2)}} \tag{10}$$

The denominator contains the same trinomial as before, but there is now another, much lower, critical frequency trinomial in the *numerator*. We have a damped *phase-advancing* integration*, i.e. a transfer function whose phase

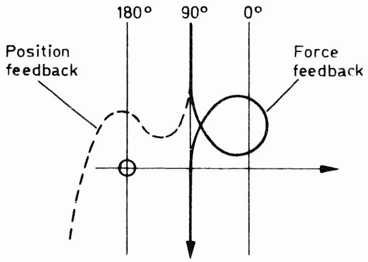


Figure 9.5

displacement never exceeds 90°, while for the position feedback system the denominator trinomial causes the phase displacement to reach 270°.

The equivalent mass of the load, therefore, does not have the detrimental effect which it had in the case of the position feedback system.

Two conclusions of practical importance may be drawn from this result:

(1) *The experimental investigation of the dynamic performance of a pressure-feedback system may be made under no-load conditions.* In the limit, it may be made even without the ram, so long as it is replaced by a receiver of similar volume, shape and elasticity. This fact is confirmed by practical results.

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* For the normal, fairly low values of friction, the ratio of the two trinomials adds a loop to the vertical straight line which represents the integration on the Nichols chart. This loop is situated on the right of the vertical line representing the integration (see *Figure 9.5*), since the angular frequency, $\sqrt{r/m}$, which reduces the amplitude and advances the phase (numerator), is less than the frequency $\sqrt{(r+r_h)m}$ which increases the amplitude and retards the phase (denominator).

(2) *The theoretical investigation of pressure-feedback systems should be pursued further than for position systems*, so as to check for the possibility of instability at very high frequencies arising from physical phenomena unnoticeable in position feedback systems because their effects are filtered by the presence of the critical frequency ω_c.

9.4.3. PERFORMANCE REQUIREMENTS. CONDITIONS OF COMPATIBILITY

The main performance requirements are concerned with maximum force and velocity, static and dynamic accuracy, response speed and stability.

9.4.3.1. *Maximum force and maximum velocity*

The area S of the ram is determined by the maximum pressure, P_M, and the maximum force required, F_M:

$$S = \frac{F_M}{P_M}$$

The maximum flow, Q_M, is determined by the area S of the ram and the maximum velocity required $(\mathrm{d}z/\mathrm{d}t)_M$:

$$Q_M = S\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)_M$$

But the flow through the servo valve is proportional to the square root of the pressure drop, ΔP, across the valve.

To define the servo valve, its nominal maximum flow, Q_{MN}, must be defined, i.e. its maximum flow under the no-load pressure difference, ΔP_N, which is normally half the supply pressure P_1. Therefore, if P_i and P_j are the maximum and minimum pressures under which maximum velocity is required, Q_{MN} will be given by the larger of the two quantities

$$Q_M\sqrt{\frac{P_1}{2(P_1-P_i)}} \quad \text{and} \quad Q_M\sqrt{\frac{P_1}{2P_j}}$$

9.4.3.2. *Static accuracy*

The major part of the pressure error obtained originates from the pressure-sensing device and cannot therefore be decreased by improving the servo system. The best that can be done is to keep the part of the pressure error due to the direct chain (mainly from the servo valve) small in relation to the part due to the pressure detector. As in Section 9.3, the deficiencies of the servo valve, dead zone and hysteresis, can be evaluated in terms of the input current or, better, as a percentage of the maximum input current:

$$s = \frac{\Delta i_s}{\Delta i_M}$$

The output error due to the servo valve is given by

$$\Delta P k K_1 = \Delta i_s$$

If ΔP_s is the maximum allowable static error, the required condition is

$$kK_1 \geqslant s\frac{\Delta i_M}{\Delta P_s}$$

or, in terms of the nominal gain of the servo valve, $K_{2N} = Q_{MN}/\Delta i_M$

$$\boxed{kK_1K_{2N} \geqslant s\frac{Q_{MN}}{\Delta P_s}} \tag{11}$$

Note that s is a measure of the quality of the servo valve and Q_{MN} is determined by the maximum force and velocity requirements mentioned.

9.4.3.3. *Dynamic accuracy*

When the opposing load is displaced, the servo system, having a constant input signal, should follow it. The servo valve must open to give the necessary flow, and in order to do this, it must receive a signal which can only come from the error between the actual pressure and the pressure demanded.

The equation defining this error is

$$\Delta PkK_1K_2 = S\frac{\mathrm{d}z}{\mathrm{d}t}$$

If the maximum allowable dynamic error is ΔP_D for a maximum velocity $(\mathrm{d}z/\mathrm{d}t)_M$ and a given pressure, the required condition is

$$kK_1K_2 \geqslant \frac{Q_M}{\Delta P_D}$$

or in terms of the nominal gain K_{2N} of the servo valve

$$\boxed{kK_1K_{2N} \geqslant \frac{Q_{MN}}{\Delta P_D}} \tag{12}$$

9.4.3.4. *Speed of Response*

If, *at low frequencies*, the response speed required is that of a first-order system of time constant τ, the required condition is

$$kK_1K_2K_3' \geqslant \frac{1}{\tau}$$

It should be borne in mind that the gain of the servo valve, K_2, varies with the pressure demanded, P_d, and for $P_d \neq P_1/2$ it depends on whether the valve is filling or emptying the ram.

Similar to the case of maximum velocity, the most unfavourable conditions for response must be used. If, for example, the movement required is a step in the direction corresponding to emptying the ram when the pressure P_d in it is less than $P_1/2$, the condition is

$$\boxed{kK_1K_{2N}K_3' \geqslant \frac{1}{\tau}\sqrt{\frac{P_1}{2P_d}}} \tag{13}$$

Note—The equivalent gain for a *sinusoidal* movement is affected less, since it is the result of the combination of an increasing and a decreasing gain. Its maximum value is K_{2N} for $P_d = P_1/2$. It decreases slowly at first, as P_d moves away from $P_1/2$, and then tends rapidly to zero as P_d approaches O or P_1.

9.4.3.5. *Stability*

If it were possible to make simplifying assumptions for the components of the chain, as is done for position feedback systems, i.e. $h(p) = f(p) = l(p) = 1$ (*Figure 9.4*), there would be no upper limit to the open-loop gain, $kK_1K_2K_3'$.

In practice, this is obviously not the case: $l(p)$ may generally be considered as unity and also $h(p)$ if there is no electronic compensating network on the amplifier. But, on the other hand, $g'(p)$ may be much more complex than the approximate form obtained in Section 9.4.2, and the phase displacement of the servo valve can no longer be neglected.

Consider the condition for stability when the anomalies of the function $g'(p)$ are negligible, which is very often the case. The open-loop transfer function of the servo system is

$$\frac{kK_1K_2K_3'}{pf(p)}$$

The transfer functions of servo valves are generally very regular in form; therefore, a gain margin of 6 dB is a good criterion of stability.

If ω_s is the angular frequency at which the phase lag of the servo valve is $90°$ and A_s the corresponding attenuation, the stability condition is

$$kK_1K_2K_3' \leqslant \frac{\omega_s}{2A_s}$$

The oscillations caused by this instability are approximately sinusoidal and, as has just been shown, for sinusoidal motion the maximum value of K_2 is K_{2N}. Hence the condition for stability is

$$\boxed{kK_1K_{2N}K_3' \leqslant \frac{\omega_s}{2A_s}} \tag{14}$$

9.4.3.6. *Summary and conclusions*

Table 9.2 summarizes the results of this Section.

The following conclusions may be drawn from these results.

(*i*) When designing a pressure-feedback system, it is important to avoid imposing a severe strain on the system when the force demanded tends towards its maximum or minimum value: choose a supply pressure significantly higher than maximum pressure required and use a spring to obtain zero force with a positive pressure.

(*ii*) In a position feedback system, improvement of the *dynamic performance* of the servo valve beyond a certain level is not important. This is not true for the force-feedback system, where the servo valve generally determines the condition for stability (*Table 9.2,* condition 1).

Table 9.2

	Performance	*Condition*
1	Stability	$kK_1K_{2N}K'_3 \leqslant \dfrac{\omega_s}{2A_s}$
2	Speed of response	$kK_1K_{2N}K'_3 \geqslant \dfrac{1}{\tau}\sqrt{\dfrac{P_1}{2P_2}}$
3	Static accuracy	$kK_1K_{2N} \geqslant s\,\dfrac{Q_{MN}}{\Delta P_s}$
4	Dynamic accuracy	$kK_1K_{2N} \geqslant \dfrac{Q_{MN}}{\Delta P_D}$

(*iii*) Since the open-loop transfer function has a very regular form, *considerable* improvement may be obtained by the use of *normal compensating networks* (phase advance, integral control).

(*iv*) If the critical condition is the response speed, resort can only be made to the normal methods quoted above, but if it is one of the accuracy conditions (*Table 9.2,* conditions 3 and 4), as is often the case, the following solutions should be considered:

(*a*) improving the static performance of the servo valve (value of s);

(*b*) the possibility of relaxing the maximum force and velocity requirements (value of Q_{MN});

(*c*) finally, and most important, decreasing K'_3 so that kK_1K_{2N} may be increased without affecting $kK_1K_{2N}K'_3$ and, therefore, without changing the stability.

Since $K'_3 = B/V_t$ [eqn. (10)], a reduction in K'_3 may be achieved either by increasing the internal volume of the ram or by decreasing the bulk modulus B

(more accurately, the overall apparent bulk modulus B_0, defined in Section 5.3.3, which takes into account the elasticity of the casing).

By applying these considerations, and in complete contradiction to the methods for position feedback systems, the author was able to establish a considerable improvement by replacing a short rigid tube connecting a servo valve and ram by a fairly long flexible tube. This sort of measure must, however, be made carefully since resonant oscillation of the column of liquid may occur at frequencies in the neighbourhood of ω_s (100–300 c/s in installations of normal size).

9.4.4. OIL COLUMN OSCILLATION

The phenomenon of oscillation of the column of oil in the line between the servo valve and the ram will obviously affect the performance of the system.

Consider a force-feedback system in which the servo valve is connected to the ram by a long pipe. We will investigate the difference in behaviour of the system according the whether the pressure detector is positioned at the servo valve end or the ram end.

Let the pressure at the valve outlet be P_V and that at the ram inlet, P_R. Under dynamic conditions, these pressures may differ, and the stability of the system will depend upon which one is used in the open-loop transfer function.

Determination of P_V and P_R—The following assumptions will be made:

Tubing between servovalve and ram: no elasticity; cross-sectional area, s; filled with incompressible oil of mass m and velocity $\mathrm{d}x/\mathrm{d}t$.

Ram: volume, V, filled with oil of bulk modulus B

Servo valve unit: volume V'', filled with oil of bulk modulus B

Equations

Servo valve flow equation:

$$Q = \frac{V''}{B}\frac{\mathrm{d}P_V}{\mathrm{d}t}+s\frac{\mathrm{d}x}{\mathrm{d}t}$$

Ram flow equation:

$$s\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{V}{B}\frac{\mathrm{d}P_R}{\mathrm{d}t}$$

Equilibrium of oil in the line:

$$(P_V-P_R)s = f\frac{\mathrm{d}x}{\mathrm{d}t}+m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

The elimination of x and P_R or x and P_V between these three equations gives the two transfer functions

$$\boxed{\frac{P_V}{Q} = \frac{\left(1+\dfrac{Vf}{Bs^2}p+\dfrac{Vm}{Bs^2}p^2\right)}{p\left(\dfrac{V}{B}+\dfrac{V''}{B}+\dfrac{VV''f}{B^2s^2}p+\dfrac{VV''m}{B^2s^2}p^2\right)}} \tag{15}$$

$$\boxed{\frac{P_R}{Q} = \frac{1}{p\left(\dfrac{V}{B}+\dfrac{V''}{B}+\dfrac{VV''f}{B^2s^2}p+\dfrac{VV''m}{B^2s^2}p^2\right)}} \tag{16}$$

These functions, although established by making a number of simplifying assumptions, are remarkably accurate in practice over a very wide range (see *Figure 9.6*). Both functions have the same trinomial in the denominator,

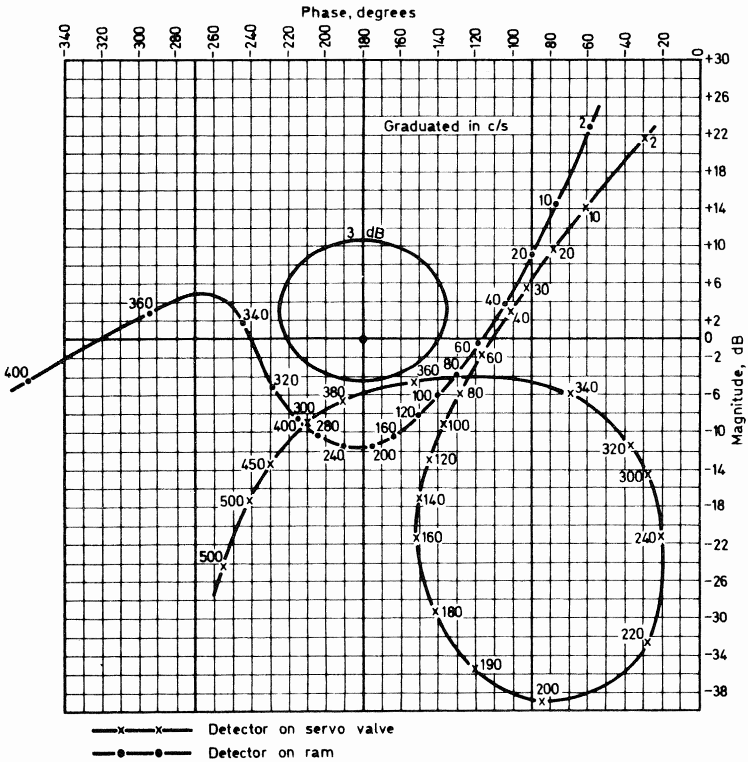


Figure 9.6. Experimental investigation of a force feedback system: effect of the position of the detector

$K = 11{\cdot}5\dfrac{\text{mV}}{\text{kg/cm}^2}$; $K_1 = 9{\cdot}5\dfrac{\text{mA}}{\text{V}}$; $K_2 = 18{\cdot}25\dfrac{\text{cm}^3\text{/sec}}{\text{mA}}$; $kK_1K_2 = 2\dfrac{\text{cm}^3\text{/sec}}{\text{kg/cm}^2}$; K_3': $= 112\dfrac{\text{kg/cm}^2}{\text{cm}^3}$

indicating the danger of vibration at high frequency, although there are ways of modifying this frequency, particularly by changing V''.

The P_R/Q transfer function (16) is reduced to the second order when the P_V/Q transfer function (15) is matched with a phase-advancing term (a loop similar to that shown in *Figure 9.5* but at much higher frequency). The result is that, if there is no compensating network and if the dangerous frequencies introduced by this new trinomial are significantly higher than ω_s, the P_R/Q transfer function has the possibility of a resonance peak only if the phase lag exceeds 180°, while the P_V/Q transfer function, matched with a phase advancer, may have a resonance peak near 180° and is therefore dangerous. *The installation of the pressure-sensing device on the ram (P_R/Q transfer function) is therefore better than on the servo valve (P_V/Q transfer function)*; cf. *Figure 9.6*.

On the other hand, if a *compensating network* is added to the amplifier, it is difficult to prevent this introducing a phase displacement. If this phase displacement is approximately compensated by the phase lead of the P_V/Q function, it is *preferable to install the pressure detector on the servo valve*. This prediction, based on fairly approximate calculations, has also been confirmed in practice.

9.5. RELIABILITY AND SAFETY

This Section does not include precise quantitative information comparable with that given in the more recent books on electronic systems. But electronic circuits normally consist of well known standard components connected in classical ways.

The reliability of a system may be estimated by the use of a few simple mathematical laws together with the results of an experimental investigation of the components themselves and of the methods of connection.

Many standard components are used in hydraulic systems: the lines and unions are always standard, the servo valves and pumps often are. But many other components have to be designed for each particular application, and their reliability, critically dependent on the radius of a curve or the abruptness of a change in section, is extremely difficult to predict, the best method possibly being that of estimating reliability of the design office and workshop responsible. being that of estimating the reliability of the design office and workshop responsible.

On the other hand, an important advantage is the fact that hydraulic systems are well adapted to *duplication of control and selector components*, functions which become more necessary when the system is made more complex, as established by recent satellite development. This aptitude is not immediately apparent, but it becomes evident after a detailed investigation of a particular circuit, as used e.g. in any modern aircraft, and it may be explained in the following way.

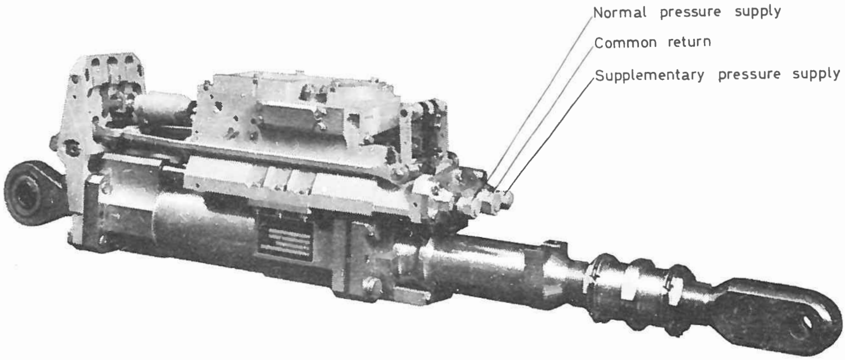
In previous Chapters, it has been shown that the so-called motor component of hydraulic servo systems is normally a linear actuator and, hence, extremely simple.

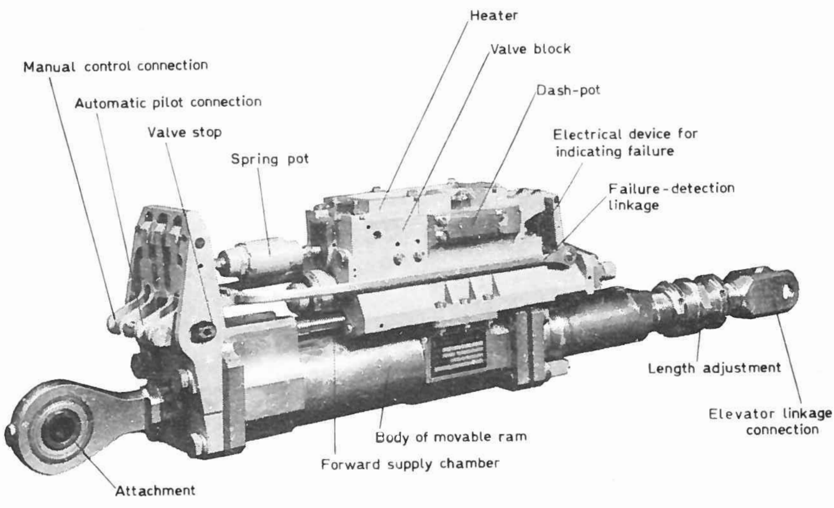
In addition, the functions of hydraulic control valves and those of the energy supply of motor-pump units are essentially completely different from the function of the actuator. They could, therefore, easily be duplicated, triplicated, etc., since controlled or automatic change-over devices are easy to design and install.

Also, general use is now being made of double rams (double-bodied rams or two rams in parallel), so that a ram which fails may be hydraulically bypassed. The result is that breakdowns which are not fail-safe are limited to the very unlikely possibility of the piston seizing up in the ram.

Perhaps the best way to illustrate this is with a specific example.

Figure 9.7 and *Plates 4–5* show a drawing and two photographs of a servocontrol designed by the author for the elevator of the Caravelle 10B aircraft. It is controlled by two identical servos mounted in parallel, each being supplied by





Plates 4 and 5. Air Equipement Servo 30076/100 controlling the elevator of the 'Caravelle' 10B3 aircraft

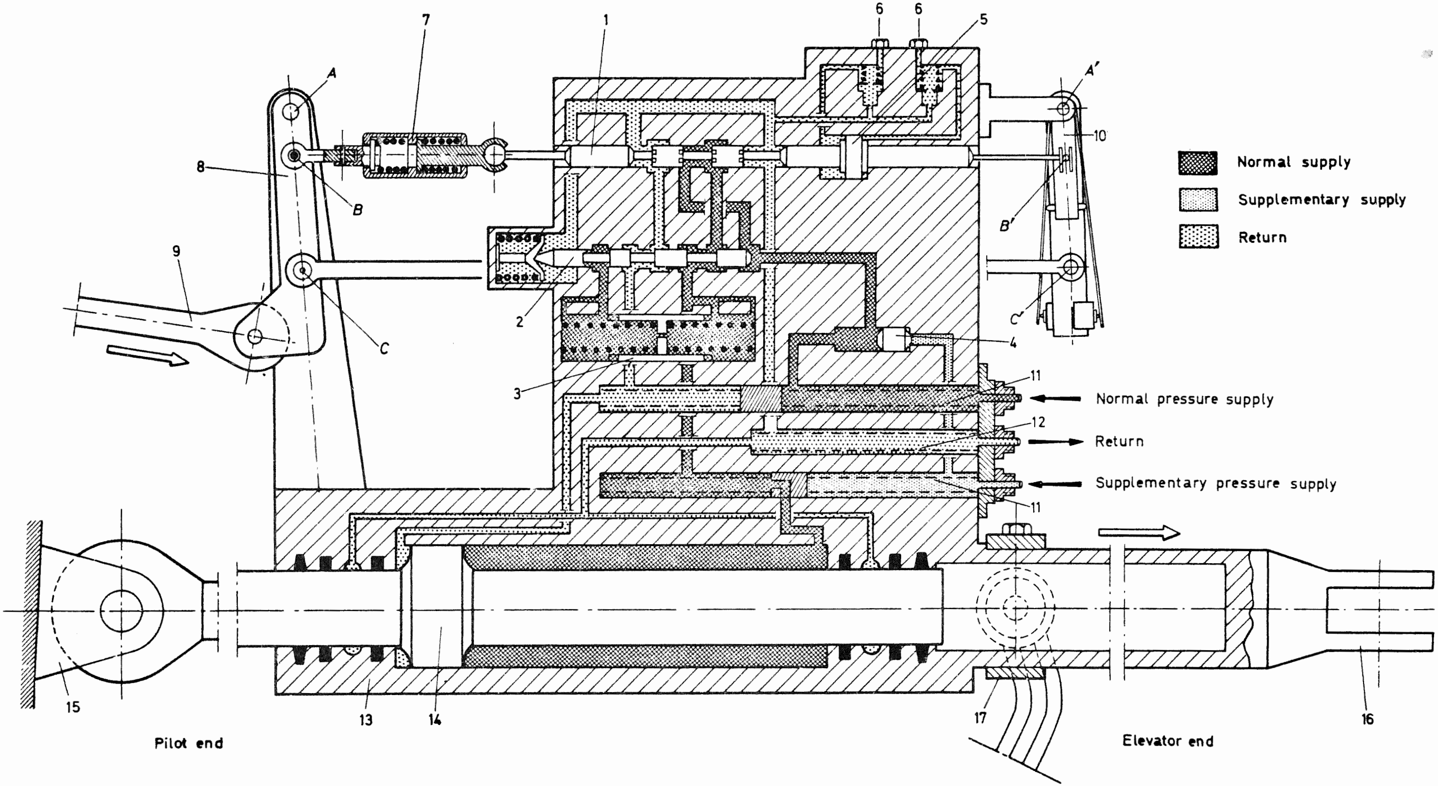


Figure 9.7. Servocontrol type 30076/100 for the 'Caravelle' 10B3 aircraft
1 Control valve; 2 bypass; 3 control surface gust damper; 4 shuttle check valve; 5 dash-pot; 6 dash-pot bleed; 7 spring-pot; 8 servocontrol lever; 9 control rod; 10 failure-indicator assembly; 11 double filter; 12 single filter; 13 body of ram; 14 piston of ram; 15 attachment to structure; 16 attachment to elevator; 17 sprung compass support.

By courtesy of Sud-Aviation Co.

two pressure sources, the shuttle valve 4 (*Figure 9.7*) automatically ensuring the selection of the highest pressure source. If both pressure sources fail, the bypass valve 2 interconnects the two chambers of the ram, so that the servo-control is freed and does not hinder the operation of the other servo which continues to give the required control.

In normal operation, the length of the control shaft, BB', is equal to that of the linkage rod, CC' and the distance AA'.

If there is a failure of the control shaft (fracture or seize-up), the spring-pot 7 causes a change in the length of BB' which initiates an electrical signal from B' to cut off the pressure supply to the servocontrol. The bypass valve 2 comes into action and the other servocontrol continues to function as before.

These are the major safety features of the design, but there are many others, such as protection against contamination (use of five filters); fracture of a joint (sprung compass support 17); air entrainment (dash-pot automatic bleed); gusts of wind when on the ground (control surface gust damper); failure of a seal (free zone between seals); ice formation, etc.

Plate 6 illustrates an electrohydraulic control system for the satellite-launching 'Diamant' rocket. The system controls the flight direction of the second stage

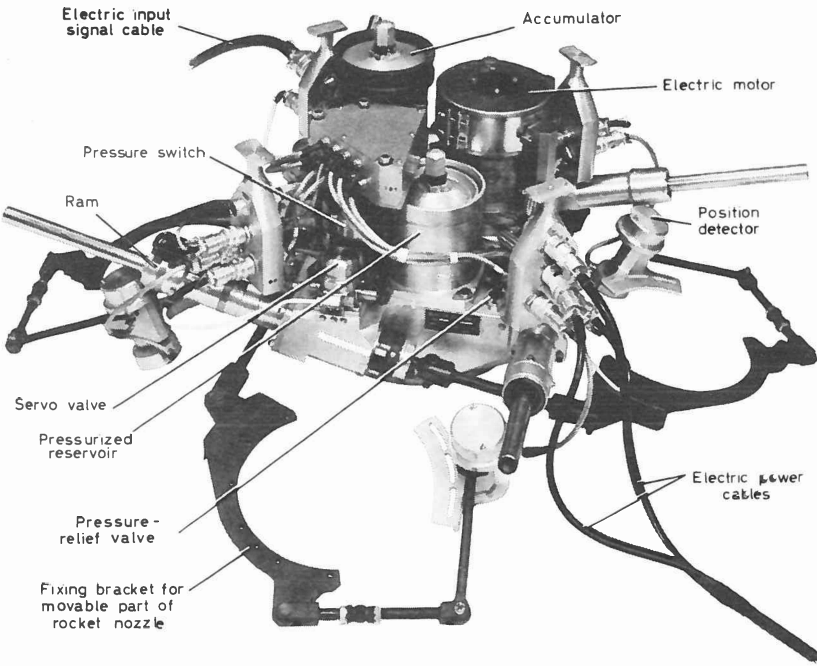


Plate 6. Electrohydraulic directional control of the second stage of the satellite-launching 'Diamant' rocket, designed and constructed by the Air Equipement branch of Ducellier–Bendix Air Equipement Co.

of the rocket by combined rotations of the four rocket nozzles. It therefore consists of four servo chains (for their functional diagram cf. *Figure 9.1*).

Note that the electronic amplifier cannot be seen, as it is separate from the assembly, situated in an instrumentation compartment where temperature and vibration conditions are less severe; that the system contains its own integrated hydraulic generating system: electric motor, hydraulic pump (not shown), high-pressure accumulator, low-pressure reservoir, valves and various control components; and that the assembly is compact and without pipework. It is delivered filled with oil and is fixed in the centre of the base of the rocket, between the nozzles. The curved brackets at the front, to the right and just visible on the left, connect the hydraulic rams to the rocket nozzles. Each ram is capable of rotating its nozzle at a speed of 3·5 rad/sec against a resisting torque of 40 m.kg which represents a power of 1·40 W. The time constant of each chain is about 10 msec.

PART III

10

APPLICATIONS*

10.1. LAMINAR AND TURBULENT FLOW (cf. Chapter 1)

An aircraft undercarriage is lowered by means of a hydraulic ram with an effective area $S = 20$ cm^2 and a travel $z = 20$ cm.

The ram is supplied with standard hydraulic fluid FHS1 of density 0·865 g/cm^3 from an accumulator of mean pressure 150 kg/cm^2 by a pipe of 4 mm internal diam., 4 m long in which there are 2 normal banjo unions, 2 abrupt elbows and an electrically controlled valve with coefficient $\xi = 3$. The external force opposing the ram has a mean value $F = 1{,}000$ kg.

(*i*) Find the time t_1 for lowering the undercarriage during static tests made in a workshop at 20°C.

(*ii*) Find the time t_2 for lowering the undercarriage after a low-speed high-altitude flight has reduced the temperature of the oil in the accumulator to -25°C.

(*iii*) If $\frac{1}{3}$ of the length of 4 mm pipe is replaced by 3 mm pipe, find the new lowering times t_1' and t_2'.

(*iv*) Find the diameter of pipe necessary to ensure that the lowering time is < 2 sec at temperatures above $T = -40$°C.

10.2. PNEUMATIC STORAGE OF ENERGY (cf. Chapter 2)

Compressed-air accumulators are used as sources of energy for certain self-supporting hydraulic systems with only a short operational life. The air can act on a piston which displaces the oil (oleopneumatic accumulator) or can bring a motor-pump unit into action. A comparison is to be made between these two methods of establishing the power stage of the pilot chain of a missile.

(*i*) Find the weight per unit volume of an accumulator compressed to a pressure P_1 as a function of the allowable tensile stress t and the specific weight d of the metal, if it is (*a*) spherical, (*b*) cylindrical and fairly long.

In order to allow for the junctions, unions, diaphragm or piston and supports, overestimate the weight by 50 per cent for the spherical accumulator and 80 per cent for the cylindrical one.

Data—Material: aluminium alloy; $P_1 = 200$ kg/cm^2; $t = 15$ kg/mm^2; volume $V_1 = 1$ l.

* These examples are presented in order both to give the reader some typical problems and some idea of the orders of magnitude of the different hydraulic parameters involved. For reasons of discretion, the numerical data of the original problems have not been reproduced. The solutions are given separately below.

(*ii*) If the minimum performance required from an oleopneumatic accumulator remains constant throughout the whole flight (so that the rams are designed for a certain final pressure, P_F) and the accumulator is filled to a given initial pressure P_1 with a total volume x and initially holding a volume of oil λx, determine the proportion λ of oil giving the accumulator of minimum volume, if the expansion of air in the accumulator is (*a*) isothermal, (*b*) adiabatic.

(*iii*) Defining the usable energy, E_u, as that which the fluid would provide if it came out of the accumulator at a constant pressure equal to the final pressure P_F, calculate $E_u = f(\lambda)$ for an accumulator of volume V_1 filled to a pressure P_1, if the expansion is (*a*) isothermal, (*b*) adiabatic.

Data: $V_1 = 1$ l.; $P_1 = 200$ kg/cm^2.

(*iv*) Now suppose that the accumulator supplies a motor-pump unit. The air from the accumulator of volume V_1 and initial pressure P_1 is first expanded to a pressure P_2 in an expansion chamber where it does no external work, then it is expanded from P_2 to pressure P_3 in a turbine of efficiency η_M. The turbine drives a pump of efficiency η_P.

If the pressures P_1, P_2 and P_3 are in absolute units, calculate the hydraulic energy provided, if the expansion in the accumulator is (*a*) isothermal, (*b*) adiabatic.

Data: $V_1 = 1$ l.; $P_1 = 200$ kg/cm^2; $P_2 = 5$ kg/cm^2; $P_3 = 1$ kg/cm^2; $\eta_M = 0{\cdot}7$; $\eta_P = 0{\cdot}8$. Compare this with the energy supplied by the oleopneumatic accumulator: $V_1 = 1$ l., $P_1 = 200$ kg/cm^2, $\lambda = 0{\cdot}3$.

(*v*) If the machine operates for a time t (sec) and requires a hydraulic power H (kg.m/sec), calculate the weight of the hydraulic installation as a function of t and H for the two cases: (*a*) oleopneumatic accumulator; (*b*) accumulator and motor pump unit.

Assume adiabatic expansion in the accumulators; initial pressure, $P_1 = 200$ kg/cm^2; weight of accumulators (without oil) $= 1$ kg/l.; filling coefficient of the oleopneumatic accumulator, $\lambda = 0{\cdot}3$; density of the oil $= 0{\cdot}865$ g/cm^3; initial temperature of the accumulator $= 15$°C; weight of the different parts (in kg) as a function of H (in kg.m/sec): rams and accessories: case (*a*) $2 + 0{\cdot}02\,H$, case (*b*) $1{\cdot}5 + 0{\cdot}015\,H$; expansion chamber, motor-pump unit, reservoir, buffer accumulator, oil, etc.: $6{\cdot}5 + 0{\cdot}015\,H$.

Plot the curve of the weight as a function of t (from 10 to 100 sec) for $H = 50$ and 500 kg.m/sec (which correspond to flows of 25 and 250 cm^3/sec under a pressure of 200 kg/cm^2.

10.3. USE OF HYDRAULIC CHARACTERISTICS (cf. Chapters 3, 1 and 2)

(*i*) A jet engine contains sixteen injectors supplied by a centrifugal pump through pipes producing a negligible loss of head. The hydraulic characteristic of each injector is the same as that of an orifice of 2 mm diam. with a coefficient of loss of head $\xi = 1{\cdot}8$ (Chapter 1). The pump normally rotates at 22,000 rev/min, and at this speed the characteristic is as shown on the logarithmic coordinates in *Figure 10.1*.

Assuming that the density of kerosene is 0·8g/cm^3, the efficiency of the pump 0·5 and that the back pressure in the combustion chamber is negligible:

(*a*) determine the total flow and the pump outlet pressure;

(*b*) find the power absorbed by the pump;

(*c*) at what speed must the pump be operated to give a flow of 15,000 l./h? What is the outlet pressure? (Assume that, when the rotational speed of the pump is multiplied by a factor λ, the flow is multiplied by λ and the outlet pressure by λ^2: cf. Section 3.3.2.2).

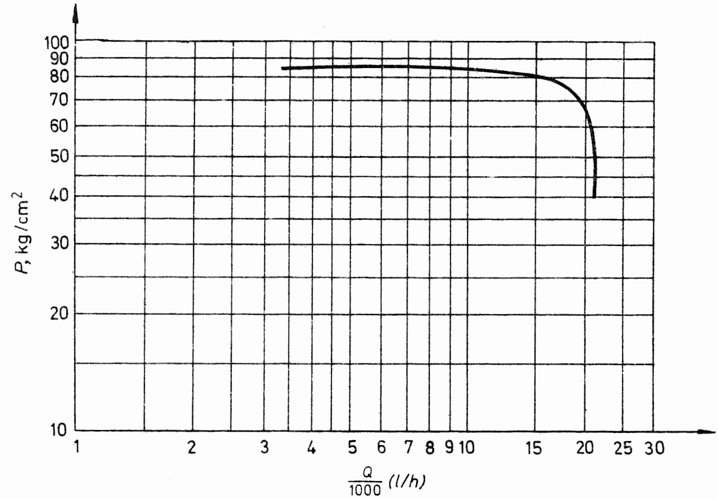


Figure 10.1

(*d*) In order to obtain flows appreciably less than those given above, for flight at high altitudes, without the outlet pressure of the pump dropping to values which would be too low to spray the fuel properly, the simple injectors must be replaced by injectors with variable cross-sectional area whose characteristics take the form $Q = K\Delta P$.

If these injectors have the same flow as the old ones at $\Delta P = 100$ kg/cm^2, find the delivery pressure and the speed of the pump at $Q = 4{,}000$ l./h.

(*ii*) A hydraulic device H whose flow is effectively independent of the supply pressure is connected to the delivery of a positive-displacement pump of fixed displacement volume and rotating at 6,000 rev/min; its flow/pressure characteristic then gives the following results:

P, kg/cm^2	0	50	100	150	170	190	210	230
Q_P, cm^3/sec	1,000	985	960	925	905	872	815	720

A bypass valve of characteristic

$$\begin{cases} Q_v = 0 \text{ for } P < P_0 \\ Q_v = K(P-P_0) \text{ for } P > P_0 \end{cases}$$

is adjusted to that when the flow through the hydraulic device H is $Q = 600$ cm^3/sec, its supply pressure is 200 kg/cm^2.

(*a*) Find the minimum slope K of the valve (in cm^3/sec per kg/cm^2) to give an increase in pressure of less than 10 per cent when Q decreases from 600 to 300 cm^3/sec. What is the pressure P_0?

(*b*) For $Q = 600$ cm^3/sec determine the pressure variations resulting from ± 20 per cent variation in the rotational speed of the pump. (Assume that the leakage flow in the pump is a function of the delivery pressure only and is zero when this is zero.)

(*c*) Find the minimum slope K' of the valve giving a variation of pressure of less than 10 per cent when Q decreases from 600 to 300 cm^3/sec and N increases by 20 per cent at the same time. What is the new pressure P_0'?

10.4. EFFECT OF HYDRAULIC FORCES ON THE CHARACTERISTICS OF A SERVO VALVE (cf. Chapters 4, 1 and 8)

Consider a standard servo valve corresponding to the diagram in *Figure 8.6* and suppose that it supplies a symmetrical equal-area ram. Let

P_1 = supply pressure = 200 kg/cm^2
P', P'' = pressures in end chambers of control valve (input to the valve, kg/cm^2)
d = density of the liquid = 0·86g/cm^3
ξ = restriction coefficient of the constrictions in the valve (cf. Section 1.4.1.2) = 1·8
D = diameter of the valve = 5 mm
j = diametral clearance between spool and sleeve = 2μ
S = effective area of ram = 6 cm^2
e = electrical input, V
Q = flow through the servo valve, cm^3/sec
G = Q/e nominal gain of the servo valve = 50 (cm^3/sec)/V for $P_1 = 200$ kg/cm^2, no load on the ram
G_1 = $P'-P''/e$ gain of the first hydraulic stage = 5 (kg/cm^2)/V

Calculate

(*i*) the valve opening necessary to give the ram a displacement speed of 0·5 m/sec when unloaded

(*ii*) the hydraulic force F_H on the valve spool

(*iii*) the restoring stiffness, R, of each of the valve springs

(*iv*) the gain, G', of the servo valve as a function of the load L on the ram; take $L = 600$ kg

Compare this with the gain of a servo valve in which the hydraulic force is negligible.

10.5. DYNAMIC PERFORMANCE OF A PRESSURE-RELIEF VALVE (cf. Chapters 5 and 6)

We will consider the pressure-relief valve shown in diagrammatic form in *Figure 10.2* and examine its dynamic behaviour in a given circuit at a given equilibrium point, when

x = valve opening ($x = 0$ when valve is closed, $x = x_0$ at equilibrium position)
$s = kx$ = cross-sectional area through which the liquid has to flow ($s_0 = kx_0$)
D = nominal valve diameter
2α = total conical angle of valve face
r = stiffness of the spring R
f = coefficient of viscous friction
m = mass of movable parts
F_0 = initial compression force in the spring (at $x = 0$)
P_1 = upstream pressure ($P_1 = P_{1_0}$ for $x = x_0$)
Q = flow through the valve ($Q = Q_0$ for $x = x_0$)

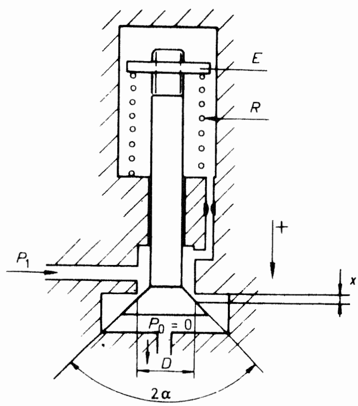


Figure 10.2

Assume that the back pressure is atmospheric ($P_0 = 0$); the hydraulic force exerted by the liquid on the valve is of the form $F_H = SP_1 - \dot{K}sP_1$ (cf. Chapter 4: 'static' force SP_1 and dynamic closing force KsP_1), and that the impedance of the upstream circuit can be defined by the expression

$$\Delta P_1/P_{10} = -(\Delta Q/Q_0)\frac{A}{1+\tau p}$$

which signifies that an increase of flow in the valve corresponds to a decrease of pressure P_1 (coefficient A) and that this decrease acts with a certain time lag (time constant, τ). The coefficient A can easily be determined from the Q, P characteristics of the upstream circuit.

(*i*) Draw the block diagram and calculate the basic transfer functions, taking the input as the variation of the initial compression force in the spring (ΔF_0

controlled* by the adjusting screw E); the output as the variation of the upstream pressure, ΔP_1, and the intermediate variable as the variation of the lift (Δx).

(*ii*) Evaluate the open-loop gain, if the compression in the spring increases by 10 per cent between $x = 0$ and $x = x_0$, and the pressure P_1 changes from P_{1_0} to $1{\cdot}5\,P_{1_0}$ when the valve is half closed: $x_0 = D/40$, $\alpha = 45°$, $K = \sqrt{2}$.

(*iii*) What are the different types of instability, their causes and possible remedies?

10.6. HYDRAULIC TRANSMISSION (cf. Chapters 5 and 6)

Consider the hydraulic transmission given in diagrammatic form in *Figure 10.3*.

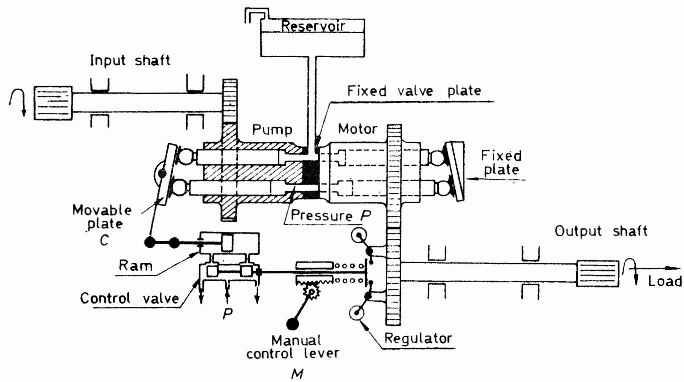


Figure 10.3

The input shaft, which is assumed to rotate at constant speed, drives a hydraulic pump of variable displacement volume which is proportional to the inclination of the control plate, C. This pump supplies a hydraulic motor which drives the output shaft through the intermediate reduction gear. The output shaft is connected to a load of known characteristics and also to a centrifugal regulator which acts on the control valve of the ram controlling plate C. The analysis will be made at a steady-state condition, using

e = input (position of manual control lever M)

Ω = angular velocity of the hydraulic motor

Ω_L = angular velocity of the load (output)

α = ratio Ω/Ω_L

* In the analysis of the stability and sensitivity to perturbations of certain components which do not have a control parameter, and therefore no proper *input*, it is often convenient to define an imaginary input which allows the use of the diagrams and methods of analysis used for servo systems. The ease of the analysis depends upon the choice of this imaginary input. It is usually advantageous to select a regulating parameter. Here, for example, we choose the initial setting of the spring R, i.e. the adjustment of the screw E, despite the fact that this screw is inaccessible and in any case maintains a fixed position on the stem during the operation of the valve. Note, however, that the introduction of an imaginary input is not theoretically necessary. It is less useful to the Germans who are more used to block diagrams of 'regulation' than to the French, British and Americans who use 'servo' diagrams.

I_m = moment of inertia of hydraulic motor
I_L = moment of inertia of load
T = torque absorbed by the load under steady conditions ($T = F\Omega_L$)
y = opening of the control valve
H_1 = transfer function $\Delta y/\Delta e$
H_2 = transfer function $\Delta y/\Delta\Omega_L$ (for the formation of transfer functions H_1 and H_2, see *Example 10.8*)
q = flow through control valve
k = gain of control valve = q/y
S = area of control ram
l = lever arm acting on plate C
β = inclination of plate C
z = displacement of ram
$Q_P = A\beta$ = flow through the pump
d_m = displacement volume/rad of hydraulic motor
V = volume of high-pressure oil between pump and motor
P = this high pressure
$Q_f = f.P$ = leakage flow of pump and motor

Neglect the inertia of the control ram and of plate C as well as the compressibility of the fluid in the ram and assume that the mechanical efficiency of the motor is 1.

(*i*) Write the functional equations, the basic transfer functions of the components and the overall open-loop transfer function. Draw the block diagram. Give a physical interpretation of the different terms of $\Delta\Omega_L/\Delta\beta$.

(*ii*) *Numerical example*

Load (maximum conditions): rotational speed, 2,500 rev/min; power absorbed, $H_L = 0{\cdot}5$ metric h.p.; moment of inertia equivalent to a weight of 1·8 kg at a distance of 2 cm from the axis.

Oil: bulk modulus, $B = 15{,}000$ kg/cm^2

Pump: maximum inclination, $\beta_M = 27°$; flow, $Q_P = 0{\cdot}1$ l./min per $°\beta$

Ram: effective area, $S = 1$ cm^2; lever arm, $l = 5$ cm

Motor: maximum speed 10,000 rev/min; number of pistons, $n = 3$; stroke of pistons, $z = 0{\cdot}6$ cm; diameter of pistons, D, to be determined, maximum value 0·7 cm; moment of inertia, $0{\cdot}5\times10^{-3}$ kg.cm.sec^2; volume at pressure P, $V = 0{\cdot}46+\pi d_m(\text{cm}^3)$; leakage coefficient, $f = 0{\cdot}5\times10^{-2}(\text{cm}^3/\text{sec})/(\text{kg/cm}^2)$; transfer function, $H_2 = 1{\cdot}5\times10^{-3}$ cm/(rad/sec)

Calculate the piston diameter for the motor, the reduction ratio, α, and the valve gain, k for a good design having a high response speed and good stability.

10.7. DESIGN OF AN ELECTROHYDRAULIC SERVO CONTROL FOR A GUIDED MISSILE (cf. Chapters 8 and 9)

The pilot chain for a guided missile is shown diagrammatically in *Figure 10.4*.

We require to design an electrohydraulic servo-control where the *input* is the electrical signal e from the automatic pilot and the *output* is the angle of the aerodynamic control surface, β. The servo-control operates as follows (see *Figure 10.5*). The voltage ϵ, the difference between the electrical input e from the automatic pilot and the feedback signal e' from the control potentiometer, is fed to an amplifier of gain K_1. The amplified voltage, $E = K_1 e$, is applied to the servo valve which supplies the two chambers C_A and C_B of the ram with flows Q_A and $Q_B = -Q_A$, respectively (put $Q_A = -Q_B = Q$). The supply pressure is P_1, and the pressure in C_A and C_B is of the order of $P_1/2$.

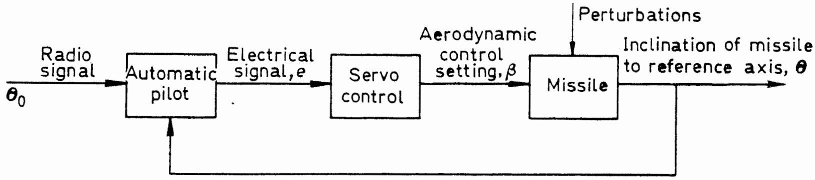


Figure 10.4

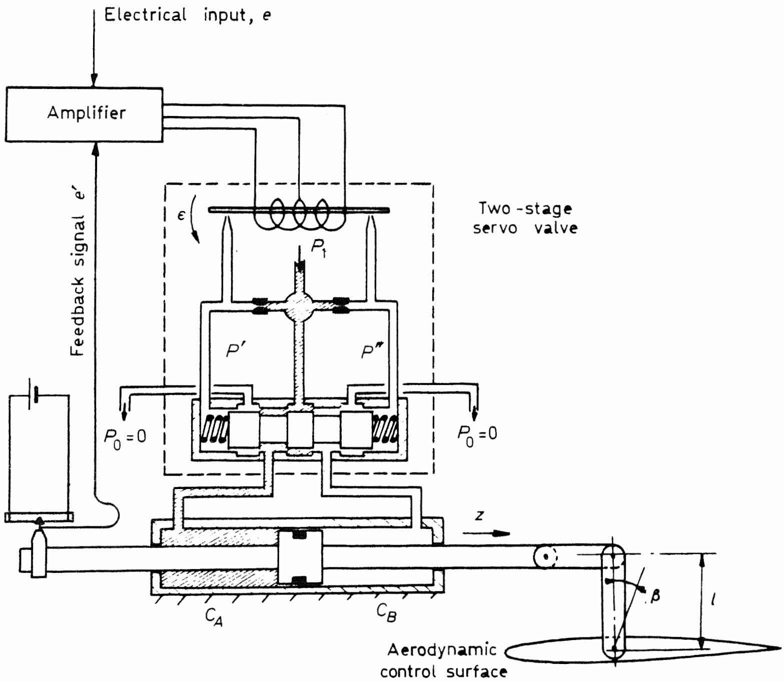


Figure 10.5. Two-stage servo valve

The transfer function of the servo valve is $Q/E = K_2 f_2(p)$ where $K_2 = 12$ (cm^3/sec)/V; $f_2(p)$ is given in *Figure 8.3b*. The hydraulic flows Q_A and Q_B fed to the chambers C_A and C_B cause a linear displacement, z, which in turn gives an angular displacement, B, of the control surface. Let $K_3 f_3(p)$ be the transfer function β/Q of the ram + aerodynamic control surface assembly.

Numerical data

Aerodynamic control surface: deflection, β, maximum amplitude $\beta_m = \pm 20°$; moment of inertia, I, equivalent to a weight of 1·5 kg at 10 cm distance from the axis; aerodynamic restoring couple, $R = 7{,}170$ cm.kg/rad deflection

Ram: length of lever to control surface, l cm; effective area: S cm^2; displacement, z cm, maximum $\pm z_M$; effective half-volume, $V_e = Sz_M$; total half-volume, $V_t = k_v V_e$ (cf. Section 7.3.3); power coefficient, k_s (assume $k_s = 1{\cdot}1$, $k_v = 1{\cdot}5$); supply pressure, $P_1 = 200$ kg/cm^2; bulk modulus of the oil, $B = 15{,}000$ kg/cm^2; pressure in chambers C_A and C_B, P_A and P_B, respectively.

Electrical input: e amplitude $\pm$ 12 V

Required closed loop performance: gain, K, 20/12 degrees deflection/V input; at 30 c/s for inputs not greater than $\frac{1}{10}$max: maximum phase difference 45°; attenuation assumed to be between 0 and 10 dB

(*i*) Draw the block diagram of the servo-control.

(*ii*) From static considerations, determine the dimensions of the ram (calculate the values of S, z_M, V_e and V_t as functions of the parameter l, neglect the effect of the inclination of the control surface lever, i.e. take $\cos\beta = 1$.

(*iii*) Having dimensioned the ram, determine the transfer function, $\beta/Q = K_3 f_3(p)$, of the ram control surface assembly, for small movements near $\beta = 0$, as a function of the parameters which define the ram: S, V_t, l, of the parameters R and I which define the load, and B. Calculate the numerical value of this transfer function.

Assume that the mass of the piston is negligible compared to that of the control surface; in the compressibility term, the variations of V_t can be neglected; the pressures P_A and P_B remain in the neighbourhood of $\frac{1}{2}P_1$.

(*iv*) Consider the following points with respect to the transfer function β/Q:

(*a*) Has the choice of the length l, which was obtained from static considerations of the ram (k_v and k_s constant), any effect on the coefficients of the transfer function? If so, which? What conclusion should be drawn?

(*b*) *For a given size ram*, what is the effect of a variation of R on the coefficients of the transfer function? For example, multiply R by 2 or divide by 10. What conclusion should be drawn?

(*c*) Suggest a simplified form of the transfer function.

(*d*) Find the resonant frequency and show the effect of variations in β and I on it, express it as a function of the parameters given in the problem (not as a function of the dimensional parameters of the ram) and show the effect of variations in P_1, k_s, k_v and R.

(*v*) Determine amplitude gain, K_1, necessary to obtain the dynamic performance required: (*a*) by an approximate calculation starting from very approximate values of the transfer functions; (*b*) on Nichols chart assuming a damping of the ram + control surface assembly represented by the reduced coefficient $\xi = 0{\cdot}2$.

(*vi*) If the control potentiometer has 500 turns and the dead zone of the servo valve extends to 0·05 V, find the order of magnitude of the accuracy of β that can be achieved. What can be done to double this accuracy?

10.8. SPEED REGULATOR FOR AN ENGINE (cf. Chapter 6)

1. UNCOMPENSATED REGULATOR

Operation (see *Figure 10.6*)—The control valve spool, V, is in equilibrium under the action of the forces F_M, from the centrifugal masses M, and F_R, from the spring R which depends on the setting of the pilot's control lever. The valve V supplies oil to the ram, R, whose piston is directly attached to the valve regulating the flow Q of fuel to the engine. A variation in Q causes a change in the rotational speed of the engine. These variations are detected by the centrifugal masses M which are driven by the engine through an intermediate reduction gear of ratio 1 : 10.

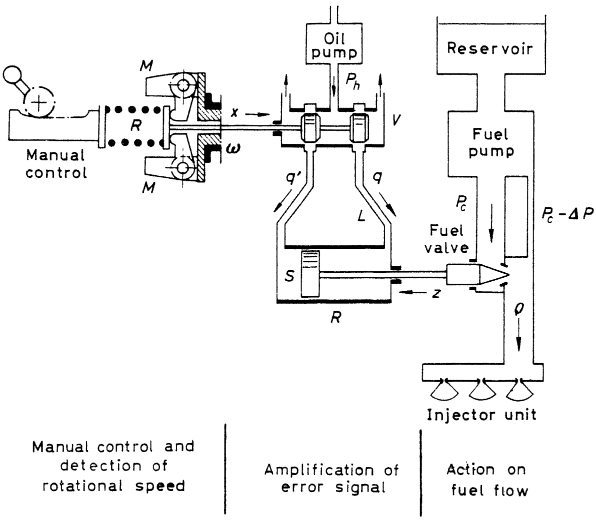


Figure 10.6

This problem concerns the dynamic performance of the regulator near a steady-state operating point. It is thus convenient to make the following change for each variable X (of value X_0 at the steady operating point)

$$\Delta X = X - X_0$$

and to make all necessary linearizations on ΔX.

The block diagram is shown in *Figure 10.7*, where the variations are:

Δu in the position of the pilot's control lever

$\Delta \omega$ of the rotational speed of the regulator

$\Delta \Omega$ of the rotational speed of the engine

Δx in the position of the valve spool (positive direction shown in diagram), composed of two parts, Δx_1 due to Δu, Δx_2 due to $\Delta \omega$

Δq of the flow of oil in the line L (positive direction shown in diagram)

Δz in position of fuel valve (positive direction shown in diagram)

ΔQ of fuel flow to the engine

The transfer functions, F_1, F_2, F_4, F_5 and F_6, must be established from the technical and experimental data given and F_3 determined so that the system is stable.

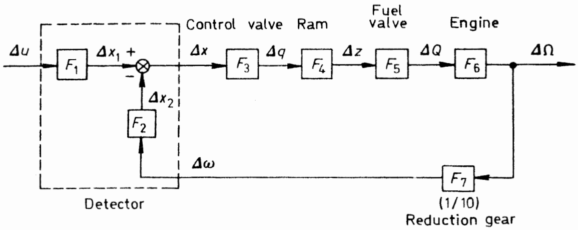


Figure 10.7

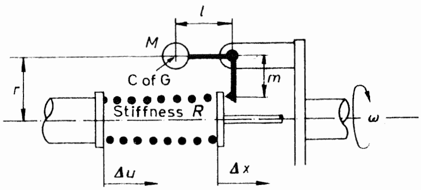


Figure 10.8

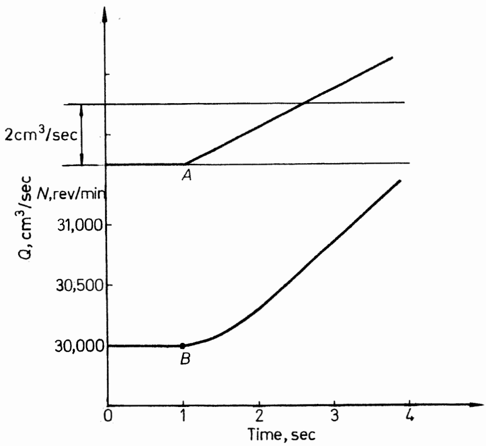


Figure 10.9

(*i*) *Estimation of F_1 and F_2*—(*a*) Calculate $\Delta x = F_1\Delta u - F_2\Delta\omega$ by linearizing the equation of equilibrium for the valve spool at $\omega = \omega_0$ (see *Figure 10.8*); (*b*) give the conditions for static stability of the detector.

Number of centrifugal masses, 2; weight of each: 20 g; $R = 12$ kg/cm, $\Omega_0 = 3{,}000$ rad/sec, $r_0 = 1{\cdot}2$ cm, $l = 1$ cm, $m = 0{\cdot}7$ cm.

(*ii*) *Estimation of F_4*—Assume that the mass and that the resisting force of the valve are negligible compared with the maximum thrust from the ram; the diameter of the ram piston is 2 cm (area S).

(*iii*) *Estimation of* F_5—If the loss of head, ΔP, through the valve is maintained constant at 20 kg/cm^2 by a special regulating device, and if the cross-sectional area of the valve restriction can be taken as being directly proportional to the displacement according to the expression $S_R = kz$ where $k = 0{\cdot}53 \times 10^{-2}$ cm, calculate $F_5 = \Delta Q/\Delta z$, given that the density of kerosene is 0·8 g/cm^3 and taking $\xi = 1{\cdot}8$ (cf. Section 1.4.1.2).

(*iv*) *Determination of* F_6—The curves of *Figure 10.9* show the simultaneous values of the fuel flow and the rotational speed as a function of time, as fuel is injected into the engine. Find F_6 by graphical construction based on simple mechanical considerations of the engine.

(*v*) *Determination of* F_3—The control valve spool constitutes a variable parameter, since its dimensions can be adjusted to suit the design. We need to determine $F_3 = \Delta q/\Delta x$ so that the regulator is stable (no amplitude rise in the closed loop).

II. REGULATOR WITH HYDRAULIC COMPENSATING NETWORK

A simple compensating network (*Figure 10.10*) gives a considerable improvement in the performance of the regulator. It consists of a piston A of area σ attached to the control valve spool and subjected on each side to pressures P_m and P_v, upstream and downstream, respectively, of piston B of area Σ which is mounted in series in line L. The two sides of B are connected by an orifice O, through which the flow is assumed laminar, with a flow coefficient $C = q/(P_m - P_v)$. B is restored to its mean position by two springs of total stiffness K.

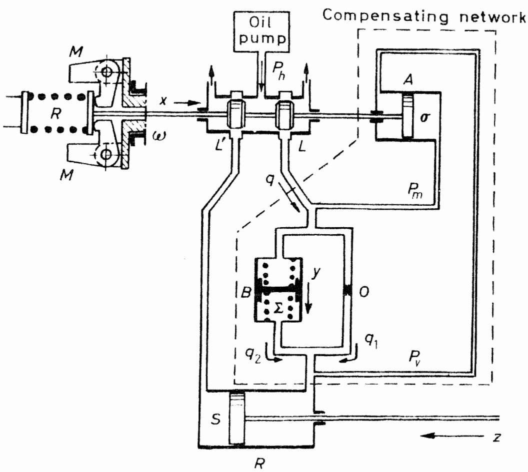


Figure 10.10

The flow coefficient C is sufficiently high to ensure that the loss of head $(P_m - P_v)$ produced in line L by the orifice O is always small compared with $P_h/2$. Under these conditions, the flow q is not affected by the presence of piston B in line L, and the action of the compensator is confined to the introduction of a supplementary force

$$\Delta F_c = -\sigma\,(\Delta P_m - \Delta P_v)$$

(*i*) If the flow Δq is made up of Δq_1, passing through the orifice O, and Δq_2 corresponding to the displacement of the piston B (Δy), calculate $(\Delta P_m - \Delta P_v)$ as a function of Δq and then as one of Δx*.

(*ii*) Calculate the new values F_1' and F_2' of F_1 and F_2 and express them in the form $F_1' = A_1 F_1$ and $F_2' = A_2 F_2$.

(*iii*) Comment on the compensating network in the light of the values obtained for A_1 and A_2.

(*iv*) Given that

$$F_3 = 1{,}150\ \frac{\text{cm}^3/\text{sec}}{\text{cm}}, \qquad \Sigma = 3\ \text{cm}^2$$

$$\sigma = 0{\cdot}8\ \text{cm}^2 \qquad K = 9\ \text{kg/cm}$$

$$C = 2\ \frac{\text{cm}^3/\text{sec}}{\text{kg/cm}^2}$$

calculate the numerical value of the transfer function. Is the regulator stable? Would it be with the same value of F_3 but without the conpensating network?

10.9. TWO-STAGE ELECTROHYDRAULIC SERVO VALVE (cf. Chapters 5, 6 and 8)

Consider the two-stage servo valve shown in *Figure 8.2* and, schematically, in *Figure 10.11*.

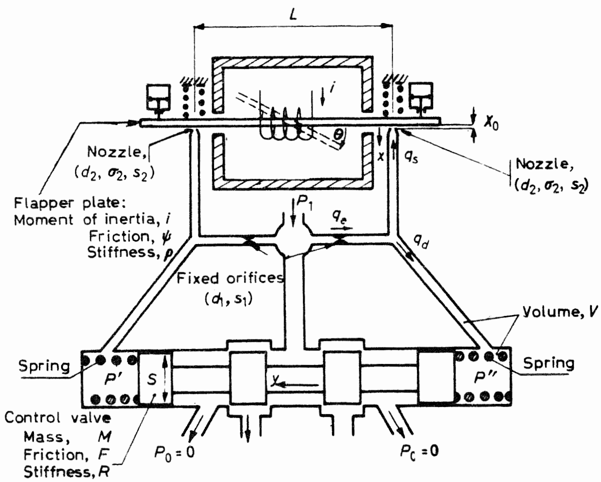


Figure 10.11

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* To make the expressions less cumbersome, put

$$F_1 = \frac{R}{R-B}, \quad F_2 = \frac{A}{R-B}, \quad a = 1 + \frac{F_3\,\sigma}{C\,(R-B)}$$

Notation and numerical values:

Hydraulic fluid

Bulk modulus	$B = 15{,}000\ \text{kg/cm}^2$
Supply pressure	$P_1 = 150\ \text{kg/cm}^2$
Density	$\rho^\star = 0{\cdot}86 \times 10^{-6}\ \dfrac{\text{kg.sec}^2}{\text{cm}^4}$

Electrical stage

Torque/current gain	$K_1 = 25\ \text{kg.cm/A}$

First hydraulic stage

Angle of rotation of flapper plate from its equilibrium position (zero input)	θ
Rotational stiffness coefficient of flapper plate	$\rho = 30\ \text{kg.cm}$
Rotational viscous friction coefficient of plate	$\psi = 10^{-2}\ \text{kg.cm.sec}$
Moment of inertia of flapper plate	$j = 2 \times 10^{-6}\ \text{kg.cm.sec}^2$
Distance between nozzles	$L = 2{\cdot}5\ \text{cm}$
Deflection of flapper plate in direction of jet	$x = \theta L/2$
Distance between nozzle and flapper plate at equilibrium position	X_0
Diameter of nozzles (with σ_2 = their cross-sectional area and s_2 = area of fluid jet)	$d_2 = 0{\cdot}07\ \text{cm}$
Coefficient of loss of head of the nozzles (Section 1.4.1.2)	$\xi_2 = 1{\cdot}2$
Diameter of fixed orifices (area s_1)	$d_1 = 0{\cdot}015\ \text{cm}$
Coefficient of loss of head of fixed orifices	$\xi_1 = 1{\cdot}7$
Pressure of fluid between fixed orifice and nozzle: for $i = 0$	$P_2 = P_1/4$
Pressure of fluid between fixed orifice and nozzle: for $i \neq 0$	$\begin{cases} P'' = P_2 + P \\ P' = P_2 - P \end{cases}$
Volume of fluid at pressure P' or P''	$V = 1\ \text{cm}^3$
Flow through hydraulic potentiometer at	$\begin{cases} \text{inlet (orifice) } q_e \\ \text{outlet (nozzle) } q_s \\ \text{equilibrium } q_0 \end{cases}$

Second hydraulic stage

Diameter of control valve (area S)	$D = 0{\cdot}475\ \text{cm}$
Stiffness of each valve spring	$R/2 = 75\ \text{kg/cm}$
Viscous friction coefficient of valve	$F = 0{\cdot}2\ \text{kg.sec/cm}$
Mass of movable part of valve	$M = 5 \times 10^{-6}\ \text{kg.sec}^2\text{/cm}$
Displacement of valve spool	y

Note—Neglect the dynamic hydraulic forces on the valve. To simplify the calculation, use the intermediate variables

$$\alpha = \frac{P_1 q_0}{2 P_2 (P_1 - P_2)} \qquad \text{and} \qquad \beta = \frac{L q_0}{2 X_0}$$

(*i*) Calculate the initial compression force, F_0, in the springs, the steady flow $2q_0$ in the two hydraulic potentiometers at the equilibrium position and the distance X_0 between nozzles and plate.

(*ii*) Calculate the gain $K = (y/i)_0$ of the servo valve, assuming that the displacement of the plate is negligible. Find y when $i = 12$ mA.

(*iii*) Give the functional equations.

Draw the block diagram.

Give the algebraic expression for the transfer function y/i and find its numerical value.

Give the exact value of the gain of the servo valve and compare it with the approximate value found above.

(*iv*) The valve is to be changed to a flow-controlled servo valve by removing the springs R and adding flow meters applying a couple $C_Q = H_0 Q$ to the plate (see Section 8.2.3.6).

If K_2 is the gain Q/y of the initial servo valve, what must be the value of H_0 if the static performance is to remain unchanged? Comment on the dynamic performance.

SOLUTIONS

EXAMPLE 1

(*i*) Part of supply pressure used to balance load:

$$\Delta P_1 = \frac{F}{S} = \frac{1\,000}{20} = 50 \text{ kg/cm}^2$$

Part of supply pressure used to produce the flow:

$$\Delta P_2 = 150 - 50 = 100 \text{ kg/cm}^2$$

CALCULATION OF FLOW VELOCITY AND OF t_1 ASSUMING TURBULENT FLOW

The sum of the localized losses of head [see Section 1.3, eqn. (3)], and the distributed loss of head [see Section 1.5.1, eqn. (6)] is

$$\Delta P_2 = \frac{\gamma}{2\,g}\left[\Sigma\,\xi + \lambda\,\frac{L}{D}\right] V^2$$

Assuming that the flow is turbulent and taking as values of the coefficients

for each banjo union, $\xi = 2{\cdot}5$ (Section 1.4.2.1)
for each elbow joint, $\xi = 1{\cdot}0$ (Section 1.4.2.1)
and $\lambda = 0{\cdot}025$ (Section 1.5.2)

we have $\Sigma\xi + \lambda L/D = 35$.

With lengths in m and forces in kg we have

$$V^2 = \frac{\Delta P_2}{\frac{\gamma}{2g}\left[\Sigma \xi + \lambda \frac{L}{D}\right]} = \frac{100 \cdot 10^4}{\frac{865}{19{\cdot}6} \cdot 35} = 648$$

$\therefore V = 25{\cdot}4$ m/sec

wherefrom $Q = 320$ cm^3/sec and, since $Sz = 400$ cm^3, $t_1 = 1{\cdot}25$ sec.

If the pipe has a fairly rough surface (Section 1.5.2, Note 3), t_1 becomes 1·31 sec. Using Graph G, the increase in λ in the neighbourhood of the critical Reynolds number gives $t_1 = 1{\cdot}41$ sec.

VERIFICATION OF THE EXISTENCE OF TURBULENT FLOW

We must now check that the assumption of turbulent flow was valid.

At 20°C, the kinematic viscosity of FHS1 fluid is about 30 cS (Graph A) so that

$$Re = \frac{V \cdot D}{\nu} = \frac{25{\cdot}4 \cdot 10^3 \times 4}{30} = 3{,}386$$

The flow is definitely turbulent and the above result is valid.

(*ii*) At -25°C, the viscosity of the oil is about 210 cS. If the flow remained the same, we would have

$$Re = \frac{V \cdot D}{\nu} = \frac{25{\cdot}4 \cdot 10^3 \times 4}{210} = 484$$

The flow is no longer turbulent and the calculation is not valid.

Laminar flow does not appear so soon in the connecting components as in the pipes themselves. But here, since the pipes cause 25/35 of the total loss of head, as a first approximation we can assume an identical behaviour of pipes and connections, i.e. we can replace the circuit by an equivalent pipe of length $4 \times (35/25) = 5{\cdot}6$ m.

Eqn. (10) of Section 1.5.3 gives

$$\begin{aligned} Q &= 2{\cdot}41 \cdot 10^6 \frac{1}{\nu w} \frac{D^4}{L} \Delta P \\ &= 2{\cdot}41 \cdot 10^6 \frac{1}{210 \cdot 0{\cdot}865} \cdot \frac{0{\cdot}0256}{560} \cdot 100 \\ &= 61 \text{ cm}^3/\text{s}, \end{aligned}$$

wherefrom

$$t_2 = 6{\cdot}55 \text{ sec}$$

If the connections are completely neglected, we get

$$Q = 85{\cdot}5 \text{ cm}^3/\text{sec and } t_2 = 4{\cdot}68 \text{ sec}$$

For a more accurate value of t_2 we can proceed by interpolation, finding the values of the loss of head necessary to give flows between 60 and 80 cm^3/sec:

Q (cm^3/sec)	60	80
V (cm/sec)	476	635
$Re = VD/\nu$	91	121
$\lambda = 64/Re$ [Section 1.5.3, eqn. (8)]	0·705	0·529
$\lambda L/D$	705	529
ξ banjo (Graph F)	14	11·5
$\Sigma\xi \simeq 4\xi$ banjo*	56	46
$\Sigma\xi+\lambda L/D$	761	575
$\Delta P_2 = (w/2g)(\Sigma\xi+\lambda L/D)V^2$	76·1	102·4

* Assuming that a group of 2 banjo joints, 2 elbows and 1 control valve is equivalent to 4 banjo joints.

By interpolation between these two values, $Q = 78$ cm^3/sec, so that

$$t_2 = 5{\cdot}1 \text{ sec and } t_2/t_1 = 4{\cdot}08$$

The drop in temperature increases the lowering time for the undercarriage by a factor of more than 4.

(*iii*) *Estimation of* t_1'—For turbulent flow, 1·33 m of 3 mm pipe is equivalent to $1{\cdot}33\times(\frac{4}{3})^5 = 5{\cdot}6$ m of 4 mm pipe (cf. Section 1.5.2, Note 2). For the purposes of calculation we can assume that we have $2{\cdot}66+5{\cdot}6 = 8{\cdot}26$ m of 4 mm pipe. Therefore

$$\Sigma\,\xi+\lambda\frac{L}{D} = 61{\cdot}5$$

$$V = 19{\cdot}2 \text{ m/sec}$$

$$t_1' = 1{\cdot}66 \text{ sec*}$$

$$Re \text{ for the 4 mm pipe} = \frac{19{\cdot}2\times10^4\times4}{30} = 2{,}560$$

$$\text{for the 3 mm pipe} = \frac{19{\cdot}2\times10^4\times(\frac{4}{3})^2\times3}{30} = 2{,}560\times\tfrac{4}{3} = 3{,}410$$

The flow is still turbulent and the value of t_1' is valid.

Estimation of t_2'—If the flow is laminar, 1·33 m of 3 mm pipe is equivalent to $1{\cdot}33\,(\frac{4}{3})^4 = 4{\cdot}21$ m of 4 mm pipe (cf. Section 1.5.3). We can, therefore, assume that we have $2{\cdot}66+4{\cdot}21 = 6{\cdot}87$ m of 4 mm pipe.

When calculating t_2 for the 4 m pipe, with the couplings introducing a negligible supplementary loss of head, the flow was 78 cm^3/sec. Since the flow is

* If we used Graph G, we would get $t_1' = 1{\cdot}8$ sec.

inversely proportional to the length of the pipe, in this case we will get approximately:

$$\frac{78 \times 4}{6{\cdot}87} = 45{\cdot}4 \text{ cm}^3/\text{sec}$$

We will again use the method of interpolation, with $Q = 42$ and $Q = 48$ cm^3/sec.

Q (cm^3/sec)	42	48
V (cm/sec)	333	381
$Re = VD/\nu$	63·5	72·6
$\lambda = 64/Re$	1·01	0·883
$\lambda L/D$	1,730	1,520
ξ banjo	19	17
$\Sigma\xi \simeq 4\Sigma$ banjo	76	68
$\Sigma\xi + \lambda L/D$	1,806	1,588
$\Delta P_2 = (w/2q)[\Sigma\xi + \lambda L/D]V^2$	88·3	101·5

Thus, by interpolation

$$Q = 47{\cdot}2 \text{ cm}^3/\text{sec}$$

and so

$$t_2' = 8{\cdot}45 \text{ sec}$$

$$\frac{t_2'}{t_1'} = 5{\cdot}10$$

The smaller-area pipe is even more sensitive to the lowering of temperature.

(*iv*) *Estimation of D*—At -40°C: $\nu = 700$ cS. If we provisionally neglect the loss of head in the connections, Section 1.5.3 eqn. (10) gives

$$D = \left(\frac{Q\nu wL}{2{\cdot}41 \times 10^6 \Delta P}\right)^{1/4} = 0{\cdot}667 \text{ cm}$$

If we assume that the connections provide a supplementary loss of head of 10/25 (which is a pessimistic estimate, as we saw above), we get

$$D = 0{\cdot}667\left(\frac{35}{25}\right)^{1/4} = 0{\cdot}726 \text{ cm}$$

A calculation using interpolation, similar to those given above, gives $D = 0{\cdot}69$ cm.

A pipe of 7 mm diam. will be sufficient to ensure a lowering time of less than 2 sec.

EXAMPLE 2

(*i*) *Weight of accumulators*—Taking D as the mean diameter, e as the thickness and assuming that e is small compared with D, we have

Type of accumulator	*Spherical*	*Cylindrical*
Thickness, e	$\dfrac{P_1 D}{4t}$	$\dfrac{P_1 D}{2t}$
Weight per unit volume (theoretical)	$\dfrac{3}{2}\dfrac{P_1 d}{t}$	$2\dfrac{P_1 d}{t}$
Weight for $V_1 = 1$ l. $P_1 = 200$ kg/cm^2 $t = 15$ kg/mm^2 $d = 2{\cdot}8$ kg/dm^3	0·56 kg	0·75 kg
Weight of real accumulator	0·84 kg	1·35 kg

(*ii*) The volumes of the ram, and therefore the total oil capacity λX, are inversely proportional to the final pressure P_F, since they are related by the equation $\lambda X = CP_F$ where C is a constant.

(*a*) *If the expression is isothermal*:

$$\frac{P_F}{P_1} = \frac{V_1 \text{ air}}{V_F \text{ air}} = \frac{X(1-\lambda)}{X} = 1-\lambda$$

so that

$$X = \frac{C}{\lambda P_F} = \frac{C}{P_1}\frac{1}{\lambda(1-\lambda)}$$

X is a minimum when $\lambda = 0{\cdot}5$.

(*b*) *If the expression is adiabatic*:

$$\frac{P_F}{P_1} = \left(\frac{V_1 \text{ air}}{V_F \text{ air}}\right)^{\gamma} = (1-\lambda)^{\gamma}$$

$$X = \frac{C}{\lambda P_F} = \frac{C}{P_1} = \frac{1}{\lambda(1-\lambda)^{\gamma}}$$

X is a minimum for $\lambda = 1/(\gamma+1)$; *so if* $\gamma = 1{\cdot}4$, $\lambda = 0{\cdot}416$.

Note—The size of a ram, and therefore its weight and volume, increases with $1/P_F$. The optimum value of λ is therefore less than that calculated above. *In practice, λ is between* 0·25 *and* 0·35.

(*iii*) The usable energy is

(*a*) *isothermal expansion*: $E_u = \lambda V_1 P_F = P_1 V_1 \lambda(1-\lambda)$ so for $P_1 = 200$ kg/cm² and $V_1 = 1$ l., $E_u = 2{,}000\,\lambda(1-\lambda)^\gamma$ kg.m.

(*b*) *adiabatic expansion*: $E_u = \lambda V_1 P_F = P_1 V_1 \lambda(1-\lambda)^\gamma$ so for $P_1 = 200$ kg/cm² and $V_1 = 1$ l., $E_u = 2{,}000\,\lambda(1-\lambda)^\gamma$ kg.m (see *Figure 10.12*).

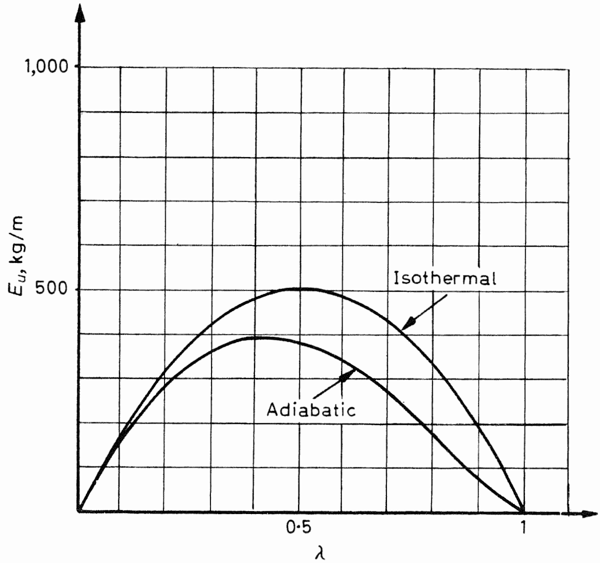


Figure 10.12

(*iv*) Call the instantaneous pressure inside the accumulator during its operation P and consider an infinitely small elemental volume, dV_1, of fluid near the outlet. When this elemental volume has come out of the accumulator, the internal pressure will be $P-dP$, with

Isothermal case:

$$\frac{dP}{P} = \frac{dV_1}{V_1}$$

Adiabatic case:

$$\frac{dP}{P} = \gamma\frac{dV_1}{V_1}.$$

Since the volume dV_1 expands from pressure P to P_2 without doing any external work and, therefore, according to the Joule–Kelvin law, without change of temperature, its volume at the inlet to the motor will be

$$dV_2 = dV_1\frac{P}{P_2}$$

and as it flows through the motor, the mechanical work done on the shaft will be:

$$\mathrm{d}E = \eta_M \frac{\gamma}{\gamma-1} P_2 \,\mathrm{d}V_2 \left[1-\left(\frac{P_3}{P_2}\right)^{(\gamma-1)/\gamma}\right]$$

$$= \eta_M \frac{\gamma}{\gamma-1}\left[1-\left(\frac{P_3}{P_2}\right)^{(\gamma-1)/\gamma}\right] P\,\mathrm{d}V_1{}^{*}$$

Thus, replacing $\mathrm{d}V_1$ by its value as a function of $\mathrm{d}P$:

Isothermal case:

$$\mathrm{d}E = \eta_M \frac{\gamma}{\gamma-1}\left[1-\left(\frac{P_3}{P_2}\right)^{(\gamma-1)/\gamma}\right] V_1\,\mathrm{d}P$$

Adiabatic case:

$$\mathrm{d}E = \eta_M \frac{1}{\gamma-1}\left[1-\left(\frac{P_3}{P_2}\right)^{(\gamma-1)/\gamma}\right] V_1\,\mathrm{d}P$$

wherefrom the equation for the total hydraulic work done,

$$E_h = \eta_P \int_{P_2}^{P_1} \mathrm{d}E$$

gives

Isothermal case:

$$E_h = \eta_M \eta_P P_1 V_1 \left[1-\frac{P_2}{P_1}\right]^{\gamma/(\gamma-1)} \left[1-\left(\frac{P_3}{P_2}\right)^{(\gamma-1)/\gamma}\right]$$

Adiabatic case:

$$E_h = \eta_M \eta_P P_1 V_1 \left[1-\frac{P_2}{P_1}\right]^{1/(\gamma-1)} \left[1-\left(\frac{P^3}{P_2}\right)^{(\gamma-1)/\gamma}\right]$$

NUMERICAL ANSWERS

Isothermal case:

$$E_h = 1{\cdot}26\,\eta_M \eta_P P_1 V_1 = 1{,}410\,\text{kg.m}$$

Adiabatic case:

$$E_h = 0{\cdot}91\,\eta_M \eta_P P_1 V_1 = 1{,}007\,\text{kg.m}$$

With the oleopneumatic accumulator, $\lambda = 0{\cdot}3$, we have

Isothermal case:

$$E_u = 0{\cdot}21 P_1 V_1 = 420\,\text{kg.m}$$

Adiabatic case:

$$E_u = 0{\cdot}182 P_1 V_1 = 364\,\text{kg.m}$$

* In the case of an adiabatic expansion, the temperature of the air which *arrives* at the pressure-reducing valve is not constant; it is the same as that of the air which comes out of the pressure-reducing valve and arrives at the motor.

The motor-pump unit gives 3·35 and 2·75 times more energy than the oleopneumatic accumulator. This more complex solution could be of interest if the weight specifications are severe or the operation of long duration.

(*v*) *Estimation of the weight* (kg) *of the installation*

		(*a*) *Oleopneumatic accumulator*	(*b*) *Motor-pump unit*
Necessary volume of accumulator, l. V_1		$\frac{H\times t}{364}$	$\frac{H\times t}{1{,}007}$
Weight of casing, w_e		$2{\cdot}75\times 10^{-3}Ht$	$0{\cdot}993\times 10^{-3}Ht$
Weight of oil in the accumulators, w_o		$0{\cdot}3\times 0{\cdot}865\ w_e = 0{\cdot}26\ w_e$	
Weight of air in the accumulators, w_α		$1{\cdot}3\times 10^{-3}\times 200\times 0{\cdot}7\ w_e = 0{\cdot}18\ w_e$	$1{\cdot}3\times 10^{-3}\times 200\ w_e = 0{\cdot}26\ w_e$
Weight of complete accumulator		$1{\cdot}44\ w_e = 3{\cdot}96\times 10^{-3}Ht$	$1{\cdot}26\ w_e = 1{\cdot}25\times 10^{-3}Ht$
Total Weight, w_t		$2+0{\cdot}02\ H+3{\cdot}96\times 10^{-3}Ht$	$8+0{\cdot}03\ H+1{\cdot}25\times 10^{-3}Ht$
For $H =$ 60 kg.m/sec		$3+0{\cdot}198\ t$	$9{\cdot}5+0{\cdot}062\ t$
	$t = 10$	4·98	10·1
	$t = 100$	22·8	15·7
For $H =$ 500 kg.m/sec		$12+1{\cdot}98\ t$	$23+0{\cdot}62\ t$
	$t = 10$	31·8	29·2
	$t = 100$	210	85

For 50 kg.m/sec, the use of a motor-pump unit leads to a smaller total weight after $t = 48$ sec; for 500 kg.m/sec, the motor-pump unit is lighter even after 10 sec total operating time.

EXAMPLE 3

(*i*) (*a*) The operating point is the intersection of the characteristic of the pump with that of the injectors. That of the latter is a straight line of slope 2, since $\Delta P = KQ^2$. We need only find one point to draw the straight line.

We can read off Graph D, for example

$$\text{for } D = 2\text{ mm and } P = 100\,\text{kg/cm}^2,\ Q = 370\text{ cm}^3/\text{sec}$$

Thus, for 16 injectors

$$Q = 0{\cdot}37\times 16\times 3{,}600 = 21{,}300\text{ l./h}$$

Using this point, we can draw the straight line which gives

$$Q = 18{,}000\text{ l./h},\ P = 75\text{ kg/cm}^2$$

(*b*) Power absorbed by the pump $= 100$ h.p.

(*c*) When the speed of the pump is changed, its characteristic is translated parallel to a vector of slope 2 (flow multiplied by λ, pressure by λ^2).

Since the characteristic of the injectors also has a slope 2, the reduced coordinates defining the operating point remain the same. We find that

$$N = 22{,}000 \times \frac{15{,}000}{18{,}000} = 18{,}350 \,\text{rev/min}$$

$$P = 75 \times \left(\frac{15{,}000}{18{,}000}\right)^2 = 52 \,\text{kg/cm}^2$$

(*d*) Characteristic of the new injectors: a straight line of slope 1 passing through the point (P = 100 kg/cm^2, Q = 11,300 l./h). The new operating point, A (Q = 4,000 l./h, P = 19 kg/cm^2).

The new rotational speed, N', of the pump will be determined graphically. We know that the characteristic of the pump at the required speed is found by translating the first one parallel to a vector of slope 2. A straight line of slope 2 passing through the new operating point A intersects the old characteristic at B: BA is the required translational vector, so that

$$N' = N\frac{Q_A}{Q_B} = 10{,}600 \,\text{rev/min}$$

(*ii*) See the construction shown in *Figure 10.13*.

(*a*) At each operating point, flow through valve = flow from pump − flow through device H.

Calling A and B the two points on the characteristic of abscissae 200 and

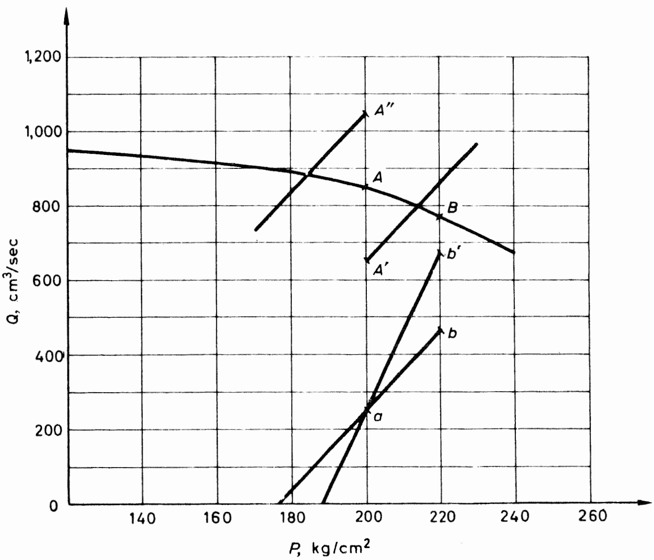


Figure 10.13

220 kg/cm^2 and subtracting 600 and 300 cm^3/sec, respectively from the corresponding ordinates of these two points, we have the points a and b of the valve limit characteristic. This is linear—the straight line ab; hence

$$K = 11 \frac{\text{cm}^3/\text{sec}}{\text{kg/cm}^2}$$

$$P_0 = 177\,\text{kg/cm}^2$$

(b) If the leakage flow is a function of pressure only, and is zero at zero pressure, the characteristic of the pump at $N \pm 20$ per cent can be found by vertical translation of amplitude ± 200 cm^3/sec. The required pressures are at the intersection of the straight line of slope K drawn through A (valve characteristic $+ 600$ cm^3/sec) with the new characteristics of the pump. To avoid having to plot these out, we can simply make a translation of the scale, i.e. find the intersections of the straight lines of slope K drawn through the points A' and A'' with the original characteristic (abscissa of A, ordinates of $A \pm 200$ cm/sec). This gives

$$P = 185 \text{ and } 215\,\text{kg/cm}^2$$

so that

$$\Delta P = \pm 7{\cdot}5 \text{ per cent}$$

(c) Same method as in (a), but b becomes b' when the ordinate is increased by 200 cm^3/sec. This gives

$$K' = 21 \frac{\text{cm}^3/\text{sec}}{\text{kg/cm}^2}$$

$$P'_0 = 188\,\text{kg/cm}^2$$

EXAMPLE 4

(i) Loss of head at each constriction: 100 kg/cm^2

Flow: 300 cm^3/sec

Velocity of flow (Graph C): 11,300 cm/sec

Cross-sectional area, $\sigma = \dfrac{300}{11{,}300} = 2{\cdot}65 \times 10^{-2}\ \text{cm}^2 = 2.65\ \text{mm}^2$

Opening:

$$y = \frac{\sigma}{\pi D} = \frac{2{\cdot}65}{15{\cdot}7} = 0{\cdot}17\ \text{mm}$$

In practice, in most servo valves, the flow passage through the control valve does not take the form of a complete annulus ($\sigma = \pi D y$) but is made up of several sectors (if there are n of width e, $\sigma = ney$). Slotted valves thus have larger openings, y; their adjustment is therefore easier but their manufacture is not.

(*ii*) Section 4.3.2.2, eqn. (19′) applies, since the clearance at the opening is small. i.e. $F = 0{\cdot}72\sigma\Delta P/\sqrt{\xi}$. Since there are two streams of flow, the total dynamic hydraulic force is

$$F_H = 2F = 2{\cdot}86 \text{ kg}$$

(*iii*) The gain of the second hydraulic stage is

$$G_2 = \frac{Q}{P' - P''} = \frac{G}{G_1} = 10 \ \frac{\text{cm}^3/\text{sec}}{\text{kg/cm}^2}$$

The value of $(P' - P'')$ necessary to give $Q = 300 \text{ cm}^3/\text{sec}$ is

$$P' - P'' = \frac{Q}{G_2} = 30 \text{ kg/cm}^2$$

The corresponding input force (static hydraulic force) is

$$F_C = \frac{\pi D^2}{4}(P' - P'') = 0{\cdot}196 \times 30 = 5{\cdot}89 \text{ kg}$$

For equilibrium of the control valve (calling F_R the force of the springs):

$$F_C = F_R + F_H,$$

$$F_R = F_C - F_H = 5{\cdot}89 - 2{\cdot}86 = 3{\cdot}03 \text{ kg}.$$

The stiffness of the two springs is given by

$$\frac{F_R}{y} = \frac{3{\cdot}03}{0{\cdot}17} = 17{\cdot}8 \text{ kg/mm}.$$

∴ Stiffness of each spring, $R = 8{\cdot}9$ kg/mm

(*iv*)
$$G' = \frac{Q}{e} = \frac{y}{e} \cdot \frac{Q}{y}$$

We must first calculate y/e, starting from the equation $F_C = F_R + F_H$

$$F_c = \frac{\pi D^2}{4}(P' - P'') = \frac{\pi D^2}{4} G_1 e = 0{\cdot}98e$$

$$F_R = 2Ry = 178y \ (y \text{ in cm})$$

$$F_H = \frac{1{\cdot}44}{\sqrt{\xi}}\sigma\Delta P = 1{\cdot}69\, y\Delta P$$

So with

$$\Delta P = \frac{P_1 - L/S}{2} = 100 - \frac{L}{12}$$

$$F_H = 1{\cdot}69\left(100 - \frac{L}{12}\right)y$$

we have

$$\frac{y}{e} = \frac{2{\cdot}8\times10^{-3}}{1-0{\cdot}4\times10^{-3}L}$$

We must now determine Q/y. Since

$$Q = \sigma\sqrt{\frac{2g\Delta P}{\xi w}}, \quad \sigma = \pi D y \quad \text{and} \quad \Delta P = \frac{P_1 - L/S}{2}$$

we have

$$\frac{Q}{y} = 10\pi D\sqrt{\frac{2g}{\xi\gamma}}\sqrt{1-\frac{L}{1{,}200}} = 17{,}800\sqrt{1-\frac{L}{1{,}200}}$$

giving finally

$$G' = \frac{Q}{e} = 50\sqrt{\frac{1-L/1{,}200}{1-0{\cdot}4\times10^{-3}L}}$$

For $L = 600$ kg, $G' = 46{\cdot}5$

If we had neglected the hydraulic force, we would have had $G' = 35{\cdot}4$ (as compared with the gain G of 50 at zero load). Good design, therefore, allows the hydraulic force to considerably reduce the variations in the gain of a standard servo valve due to variations in the load (or supply pressure P_1).

EXAMPLE 5

(*i*) All displacements and forces will be taken as positive in the direction of the arrows shown in *Figure 10.2*. The initial force, F_0, in the spring is therefore negative. The force of the spring is

$$F_R = F_0 - rx$$

$$\Delta F_R = \Delta F_0 - r\Delta x$$

The hydraulic force is

$$F_H = SP_1 - KsP_1$$

$$\Delta F_H = (S - K s_0)\Delta P_1 - KP_{10}\,\Delta s$$

$$= (S - Ks_0)\Delta P_1 - KkP_{10}\,\Delta x$$

The frictional and inertia forces are

$$-f\,\mathrm{d}x/\mathrm{d}t - m\,\mathrm{d}^2x/\mathrm{d}t^2$$

The equilibrium equation is, therefore

$$F_R + F_H - f\,\mathrm{d}x/\mathrm{d}t - m\,\mathrm{d}^2x/\mathrm{d}t^2 = 0$$

and by subtracting the equation of static equilibrium (at $x = x_0$), we have

$$\Delta F_0 + (S - K s_0)\Delta P_1 - \Delta x\,(r + KkP_{10} + fp + mp^2) = 0$$

$$(-\Delta x) = \frac{(-\Delta F_0) - (S - K s_0)\,\Delta P_1}{(r + K k P_{10} + f p + m p^2)}$$

By differentiating the flow equation, $Q = \lambda s\sqrt{P_1}$, we have

$$\Delta Q/Q_0 = \Delta s/s_0 + \Delta P_1/2P_{10} = \Delta x/x_0 + \Delta P_1/2P_{10}$$

so, substituting in the equation defining the impedance of the circuit

$$\frac{\Delta P_1}{(-\Delta x)} = \frac{P_{10}}{x_0}\,\frac{A}{1 + A/2}\cdot\frac{1}{1 + \tau p/(1 + A/2)} = \frac{P_{10}}{x_0}\,A'\,\frac{1}{1 + \tau' p}$$

from which we can draw the block diagram shown in *Figure 10.14*.

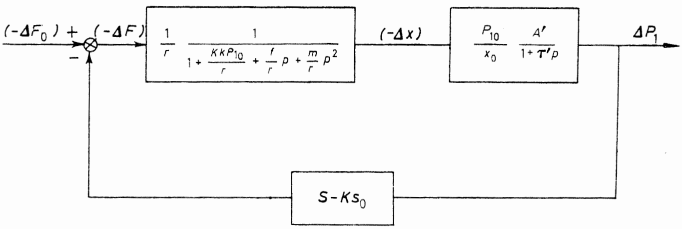


Figure 10.14

(*ii*) *Open-loop gain*

$$G = A'\frac{P_{1_0}S(1 - Ks_0/S)}{rx_0(1 + KkP_{1_0}/r)}$$

$$A' = 1$$

$$s_0 = \frac{\pi D x_0}{\sqrt{2}} \quad \text{and} \quad S = \frac{\pi D^2}{4}$$

$$\therefore \quad \frac{Ks_0}{S} = \frac{\pi D x_0}{\pi D^2/4} = \frac{4x_0}{D} = 0{\cdot}1$$

If the force of the spring increases by 10 per cent between $x = 0$ and $x = x_0$,

$$rx_0 = 0{\cdot}1F_0$$

and the equation of static equilibrium is

$$F_R + F_H = 1{\cdot}1F_0 + (S - Ks_0)P_{1_0} = 1{\cdot}1F_0 + 0{\cdot}9SP_{1_0} = 0$$

$$\therefore \quad F_0 \simeq 0{\cdot}8SP_{1_0}$$

This gives

$$SP_{1_0} = 12{\cdot}5rx_0$$

and

$$\frac{KkP_{1_0}}{r} = \frac{Kkx_0P_{1_0}}{rx_0} = \frac{Ks_0}{rx_0}P_{1_0} = \frac{Ks_0}{S} - \frac{SP_{1_0}}{rx_0} = 1{\cdot}25$$

Finally, we have

$$G = \frac{12{\cdot}5 \times 0{\cdot}9}{2{\cdot}25} = 5$$

(*iii*) *The open-loop transfer function is of third order of the form*

$$\frac{G}{\left(1 + \frac{f}{r'}p + \frac{m}{r'}p^2\right)(1 + \tau' p)}$$

We can normally distinguish two types of instability:
(*a*) The so-called static instability*, caused by the open-loop gain (with unit feedback) being less than -1. It can appear when KkP_{1_0} becomes less than $-r$, in particular for certain types of valves where K becomes negative at very small opening displacements; it is therefore advisable to return to the derivation of the equation (calculation of F_H), considering K as a function of x.

Remedy—Modify the design of the valve or increase r. When $A' = A/(1+A/2)$ becomes negative, e.g. for low values of the flow through a centrifugal pump when the pressure increases slightly with the flow ($A < 0$), there is no real remedy.

(*b*) Dynamic instability due to the presence of a time constant τ'.

Remedies—Damp the valve with a dash-pot (increasing f which is normally very small unless special precautions are taken); increase its natural frequency by decreasing m or increasing r'. Decrease the gain (by reducing the size).

EXAMPLE 6

(*i*) *Functional equations*

Regulator:

$$\Delta y = y = H_1\Delta e - H_2\Delta\Omega_L \tag{1}$$

Control valve:

$$q = ky \tag{2}$$

Ram:

$$q = Spz = Spl\beta \tag{3}$$

Pump:

$$Q_p = A\beta \tag{4}$$

* This is the instability obtained, for example, by reversing the detector connections of a classical servo system with a high gain.

Flow equation:

$$Q_p = A\beta = Q_m = d_m\,\Omega + fP + \frac{V}{B}\,p\,P$$

By subtracting the equation of flow for the steady-state conditions, we have

$$A\,\Delta\beta = d_m\,\Delta\Omega + \Delta P\left(f + \frac{V}{B}\,p\right) \tag{5}$$

Torque equation:

torque supplied by motor:

$$T_m = d_m P$$

torque absorbed by inertia of motor:

$$I_m p\Omega$$

torque absorbed by load

on the output shaft:

$$F\Omega_L + I_L p\Omega_L = 1/\alpha(F\Omega + I_L p\Omega)$$

fed back to the input shaft:

$$1/\alpha^2(F\Omega + I_L p\Omega)$$

Thus we have the equation

$$d_m P = \Omega\left[\frac{F}{\alpha^2} + \left(I_m + \frac{I_L}{\alpha^2}\right)p\right]$$

which, expressed in terms of the variations and with $I = I_L + \alpha^2 I_m$, becomes

$$d_m\,\Delta P = \Delta\Omega\left[\frac{F}{\alpha^2} + \frac{I}{\alpha^2}\,p\right] \tag{6}$$

Eliminating ΔP, we get

$$\frac{\Delta\Omega_L}{\Delta\beta} = \frac{A/\alpha\, d_m}{1 + \dfrac{Ff}{\alpha^2 d_m^2} + p\left(\dfrac{If}{\alpha^2 d_m^2} + \dfrac{VF}{B\,\alpha^2 d_m^2}\right) + \dfrac{VI}{B\,\alpha^2 d_m^2}\,p^2} \tag{7}$$

from which the block diagram (*Figure 10.15*) can be drawn.

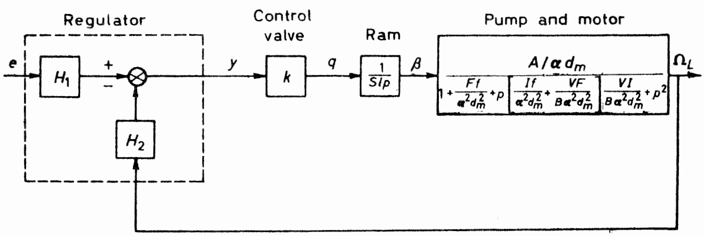


Figure 10.15

In eqn. (7) for the transfer function $\Delta\Omega_L/\Delta\beta$, we find the terms:

$A/\alpha_m d$, the steady-state gain with no leakage, obtained directly by putting:

$$A\,\Delta\beta = \Delta\Omega .\, d_m = \Delta\Omega_L\, a\, d_m$$

$fF/(\alpha d_m)^2$, denoting the decrease of gain due to the fact that the leakage flow increases with pressure and, therefore, with velocity ($T = F\Omega_L$);

$pfI/(\alpha d_m)^2$, due to the transient increase in leakage during acceleration;

$pVF/B(\alpha d_m)^2$, representing the compressibility flow necessary to pass from one steady state to the other;

$p^2 VI/B(\alpha d_m)^2$, representing the transient compressibility flow.

The open-loop transfer function is

$$FT_0 = \frac{k H_2 \dfrac{1}{Sl}\dfrac{A}{a\, d_m}}{p\left[1 + f\dfrac{F}{(a\, d_m)^2} + p\left(f\dfrac{I}{(a\, d_m)^2} + \dfrac{VF}{B(a\, d_m)^2}\right) + \dfrac{VI}{B(a\, d_m)^2}p^2\right]}$$

(*ii*) *Numerical application*

The 3 parameters which determine the dimensions of the transmission are β, d_m and α. They are not independent. Neglecting leakage, we have, to a first approximation

$$Q_p = d_m\ \Omega = d_m\, a\, \Omega_L$$

$$a = \frac{Q_p}{d_m\, \Omega_L} = \frac{100}{60}\,\beta\,\frac{1}{d_m}\,\frac{1}{2{,}500}\,\frac{30}{\pi} = \frac{\beta}{50\,\pi\, d_m} \qquad \text{with } \beta \text{ in degrees}$$

The pump rotating at constant speed provides a constant torque, $T = d_p P$. Thus, for P to be a minimum, which is important from the point of view of operation and efficiency, the displacement volume, d_p, of the pump must be a maximum. If we allow an adjustment of β up to 25°, we have

$$a = \frac{1}{2\,\pi\, d_m} = \frac{0{\cdot}159}{d_m}$$

We can now determine the extreme possible values of d_m. The upper limit is governed by the diameter of the pistons

$$d_{m_{\max}} = 3 \times \frac{\pi}{4}\,(0{\cdot}7)^2 \times 0{\cdot}6 \times \frac{1}{2\,\pi} = \mathbf{0{\cdot}110\ cm^3}$$

The lower limit is determined, for a given value of β, by the speed limit of the motor. Thus, for $\Omega_{\max}$: $Q_m = Q_p$ and $2\pi d_{m_{\min}} \times 10{,}000 = 2{,}500$

$$d_{m_{\min}} = \mathbf{0{\cdot}04\ cm^3}$$

Numerical determination of the transfer function

$$I_L = 7{\cdot}35 \times 10^{-3}\,\text{kg.cm.sec}^2$$

$$\frac{I}{\alpha_2} = I_m + \frac{I_L}{\alpha^2} = 0{\cdot}5 \times 10^{-3} + 290 \times 10^{-3} d_m^2$$

$$\frac{I}{(\alpha d_m)^2} = 0{\cdot}29 + \frac{0{\cdot}5}{10^3 d_m^2}$$

Power absorbed by the load:

$$H = 0{\cdot}5 \text{ metric h.p.} = 3{,}750 \text{ kg.cm/sec}$$

Torque:

$$T = \frac{H}{\Omega_L} = \frac{3{,}750}{2{,}500}\frac{30}{\pi} = 14{\cdot}3 \text{ kg.cm}$$

Torque coefficient:

$$F = \frac{T}{\Omega_L} = \frac{14{\cdot}3}{2{,}500}\frac{30}{\pi} = 0{\cdot}0545 \text{ kg.cm.sec}$$

$$\frac{V}{B} = \frac{(0{\cdot}46 + \pi d_m)}{15{,}000} = (0{\cdot}306 + 2{\cdot}1 d_m)10^{-4}$$

$$A = 0{\cdot}1 \frac{\text{l./min}}{\text{degree}} = 95{\cdot}5 \frac{\text{cm}^3/\text{sec}}{\text{rad}}$$

Since $f = 0{\cdot}5 \times 10^{-2}$ and $\alpha d_m = 0{\cdot}159$, this gives

$$\frac{A}{a\,d_m} = 600 \quad ; \quad k\,H_2\,\frac{1}{S\,l}\,\frac{A}{a\,d_m} = 0{\cdot}18\,k$$

$$\frac{Ff}{(\alpha d_m)^2} = 0{\cdot}0109 \text{ (negligible compared with unity)}$$

$$\frac{If}{(d\alpha_m)^2} = 1{\cdot}45 \times 10^{-3} + \frac{0{\cdot}25}{d_m^2}\,10^{-5}$$

$$\frac{V}{B}\frac{F}{(\alpha d_m)^2} = 0{\cdot}066 \times 10^{-3} + 0{\cdot}45 d_m 10^{-3} \text{ (negligible compared with } If/(\alpha d_m)^2\text{)}$$

$$\frac{V}{B}\frac{I}{(\alpha d_m)^2} = \left(0{\cdot}61 d_m + 0{\cdot}089 + \frac{1{\cdot}05 \times 10^{-3}}{d_m} + \frac{0{\cdot}153 \times 10^{-3}}{d_m^2}\right)10^{-4}$$

The open-loop transfer function is presented in the form:

$$(TF)_0 = \frac{G}{p\left(1 + 2\xi\frac{p}{\omega_0} + \frac{p^2}{\omega_0^2}\right)}$$

and ξ and ω_0 have been calculated for different values of d_m:

d_m	0·04	0·06	0·08	0·11
ω_0	207	232	240	237
ξ	0·31	0·25	0·22	0·20

As far as response speed is concerned, the Nichols chart shows a slight advantage in favour of the lowest value of d_m, but this does not compensate for the two disadvantages of having to manufacture very small pistons and cylinders for the rotor of the motor (of the order of 4 mm diam. for $d_m = 0{\cdot}04$) and of requiring a high speed of the motor. We will therefore choose $d_m = 0{\cdot}11$ cm^3, which gives $D = 7$ mm and $\alpha = 1{\cdot}45$. The corresponding Nichols chart is shown in *Figure 10.16*.

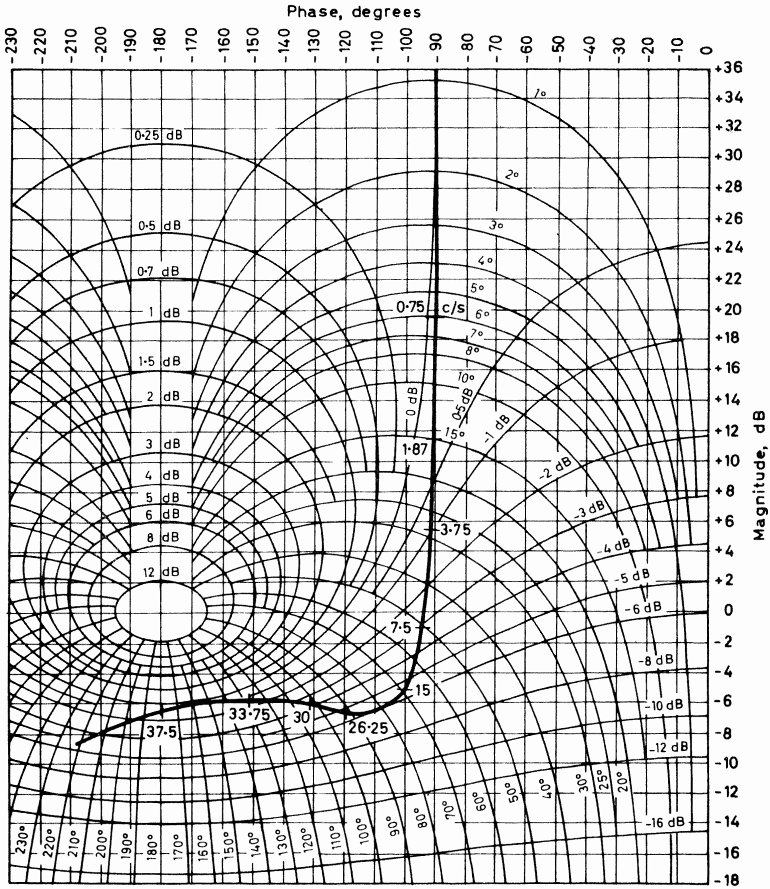


Figure 10.16. Hydraulic transmission, $d_m = 0{\cdot}11$ cm^3

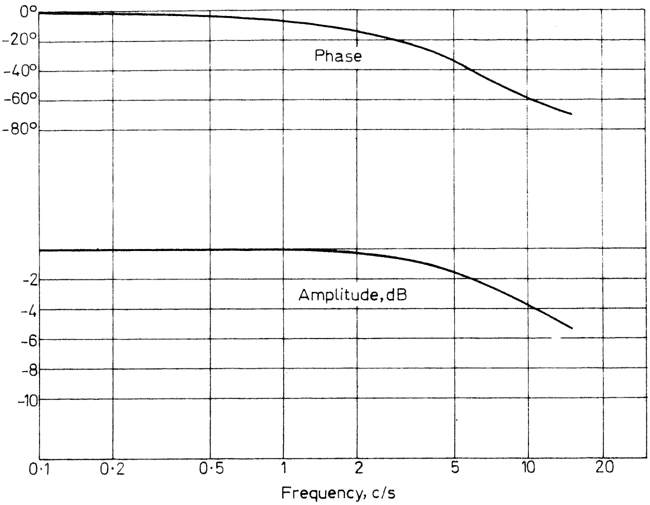


Figure 10.17. Hydraulic transmission, frequency response curves

At low frequencies the closed-loop behaves like a first-order system with a time constant of 0·024 sec (see *Figure 10.17*).

The gain required is $G = 45$. This has been shown to be 0·18 k, so that

$$k = 250 \frac{\text{cm}^3/\text{sec}}{\text{cm}}$$

Note—This calculation has been made neglecting the leakage flow. To take this into account, we find the high pressure

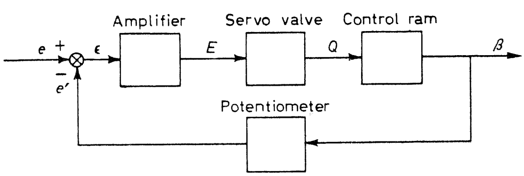
$$P = \frac{W_L}{Q_P} = 90 \text{ kg/cm}^2$$

and use this to estimate the leakage flow $Q_l = fP = 0{\cdot}45$ cm^3/sec. Thus $Q_l/Q_p = 1$ per cent, which represents a decrease of 1 per cent in α or in d_m.

EXAMPLE 7

(*i*) *Block diagram* (*Figure 10.18*)

Figure 10.18



(*ii*) *Maximum resisting torque,* $C_R = 2{,}500$ cm.kg

Maximum torque from ram $C_J = k_s C_R = 2{,}750$ cm.kg

Effective area:

$$S = \frac{C_J}{P_1 l} = \frac{k_s C_R}{P_1 l} = \frac{13 \cdot 75}{l\ (\mathrm{cm})}\ \mathrm{cm}^2$$

$$z_M(\mathrm{cm}) = l\beta_M = 0 \cdot 35\ l\ (\mathrm{cm})$$

$$V_e = Sz_M = \frac{k_s C_R}{P_1}\beta_M = 4 \cdot 81\ \mathrm{cm}^3$$

$$V_t = K_v V_e = 7 \cdot 22\ \mathrm{cm}^3$$

(*iii*) *Transfer function of the ram*

$$\frac{\beta}{Q} = \frac{1/Sl}{p\left[1 + \frac{V_t R}{2\,B\,(Sl)^2} + p^2 \frac{V_t I}{2\,B\,(Sl)^2}\right]}$$

Numerical value

$$\frac{\beta(\mathrm{rad})}{Q(\mathrm{cm}^3/\mathrm{sec})} = \frac{0 \cdot 0727}{p\left[1 \cdot 009 + \frac{p^2}{(2{,}290)^2}\right]}$$

(*iv*) (*a*) The size of l does not affect the coefficient of the transfer function; l appears in sl, and $sl = k_s C_R / P_1$. Hence, the dynamic performance cannot be improved by changing the value of l.

(*b*) R does not affect the coefficients, except in $V_t R / 2B(Sl)^2$ which is negligible compared with unity. Hence, a *given* ram has the same dynamic performance with or without the aerodynamic couple (irreversibility of servocontrols).

(*c*) The simplified transfer function is

$$\frac{\beta}{Q} = \frac{1/Sl}{p\left[1 + p^2 \frac{V_t I}{2\,B\,(Sl)^2}\right]}$$

(*d*) The resonant frequency is given by

$$\omega_C = \sqrt{\frac{2\,B\,(Sl)^2}{V_t I}}$$

which can be written

$$\omega_C = \sqrt{\frac{k_s}{k_v}\,\frac{2\,B}{P_1}\,\frac{R}{I}}$$

ω_C increases with k_s, B and R and decreases when k_v, P_1 and I are increased.

(*v*) To a first approximation, for a frequency of 30 c/s ($\omega = 189$ rad/sec), we can assume the following simplified transfer functions:

Amplifier:

$$K_1$$

Servo valve:

$$K_2 = 12 \frac{\text{cm}^3/\text{sec}}{\text{V}}$$

Ram + elevator:

$$\frac{K_3}{p} = \frac{0{\cdot}0727}{p} \frac{\text{rad}}{\text{cm}^3/\text{sec}} = \frac{4{\cdot}16}{p} \frac{\text{degrees}}{\text{cm}^3/\text{sec}}$$

Potentiometer:

$$K_4 = 0{\cdot}6 \frac{\text{V}}{\text{degrees}}$$

wherefrom we get:

Open-loop transfer function: $$\frac{K_1 K_2 K_3 K_4}{p}$$

Closed-loop transfer function: $$\frac{1/K_4}{1 + \dfrac{p}{K_1 K_2 K_3 K_4}}$$

Phase angle: A phase difference of 45° for $\omega = 189$, with

$$\frac{\omega}{K_1 K_2 K_3 K_4} < 1$$

means that

$$K_1 > \frac{\omega}{K_2 K_3 K_4}$$

$$K_1 > \frac{189}{12 \,.\, 4{\cdot}16 \,.\, 0{\cdot}6}$$

$$\mathbf{K_1 > 6{\cdot}3}$$

Attenuation: for $K_1 = 6{\cdot}3$, $A = 3$ dB, acceptable.

(*b*) *Accurate determination*

The open-loop transfer function is

$$(FT)_0 = K_1 \,.\, K_2 f_2(p) \,.\, K_3 f_3(p) \,.\, K_4$$

$f_2(p)$ can be read off the experimental frequency response curve for the servo valve, and $f_3(p)$ was determined in Section (*iii*):

$$f_3(p) = \frac{1}{p\left(1 + 2\zeta \dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right)} \qquad (\omega_c = 2{,}290)$$

This gives:

$$(F\,T)_0 = \frac{K_1 K_2 K_3 K_4}{\omega_c} \cdot f_2(p) \cdot \frac{1}{\dfrac{p}{\omega_c}\left(1 + 2\zeta\dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right)}$$

and we can plot on the Nichols chart

$$C = f_2(p)\frac{1}{\dfrac{p}{\omega_c}\left(1 + 2\zeta\dfrac{p}{\omega_c} + \dfrac{p^2}{\omega_c^2}\right)} \quad \text{(graduated in } \omega\text{)}$$

By vertical translation of the curve C, we can find the condition in which the phase difference is most severe and then we can determine the gain necessary to satisfy it:

$$A = \frac{K_1 K_2 K_3 K_4}{\omega_c} = -20 \text{ dB} = 0{\cdot}1$$

From which we get:

$$K_1 = \frac{0{\cdot}1\ \omega_c}{K_2 K_3 K_4}$$

and finally $K_1 = 7{\cdot}6$ (see *Figure 10.19*)

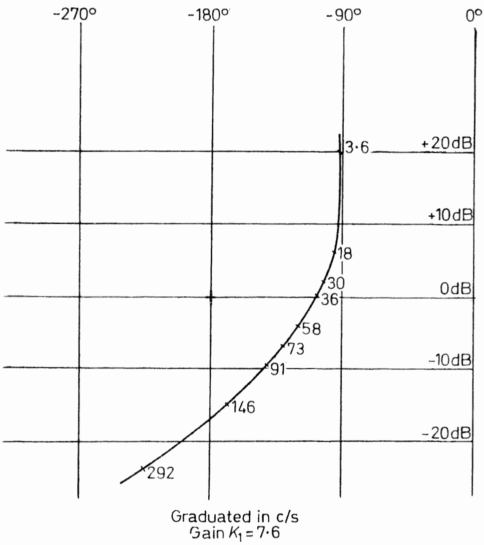


Figure 10.19

(*vi*) *Accuracy of the potentiometer*

The error in β due to the potentiometer is $4 \times 10^{-3}\beta_M$. The accuracy of the servo valve is: $0{\cdot}05/12 = 4{\cdot}17 \times 10^{-3}$. Now

$$\frac{\Delta E}{E_{\max}} = K_1 \frac{\varepsilon}{E_{\max}}$$

and since $E_{\max}$ numerically equal to $e_{\max}$

$$\frac{\Delta E}{E_{\max}} = K_1 \frac{\varepsilon}{e_{\max}} = K_1 \frac{\Delta\beta}{\beta_M}$$

The error in β due to the servo valve is

$$\frac{4{\cdot}17 \,.\, 10^{-3}}{K_1} = 0{\cdot}55 \,.\, 10^{-3}\, \beta_M$$

The accuracy depends on the potentiometer and not on the servo valve. To improve it, a potentiometer with better definition must be used or even a different method of detection altogether.

EXAMPLE 8

I. Uncompensated regulator

(*i*) *Estimation of F_1 and F_2*

Centrifugal force: $F_c = 2M\,\omega^2 r$

Force on valve spool from the centrifugal masses:

$$F_M = -F_c \frac{l}{m} = -2\,M\,\omega^2\, r\, \frac{l}{m}$$

Force from the spring:

$$F_R = F_{R0} + (\Delta u - \Delta x)\, R$$

Near the equilibrium position:

$$\Delta F_M = -4\,M\,\omega_0\, r_0 \frac{l}{m}\,\Delta\omega - 2\,M\,\omega_0^2 \frac{l}{m}\,\Delta r$$

and since

$$\Delta r = -\frac{l}{m}\,\Delta x:$$

$$\Delta F_M = -4\,M\,\omega_0 r_0 \frac{l}{m}\,\Delta\omega + 2\,M\,\omega_0^2 \left(\frac{l}{m}\right)^2 \Delta x = -A\,\Delta\omega + B\,\Delta x$$

$$\Delta F_R = (\Delta u - \Delta x)\, R$$

The equilibrium equation is written as $\Delta F_M + \Delta F_R = 0$, neglecting the inertia forces which are significant only at very high speeds.

$$\Delta x \cdot \left[R - 2\,M\,\omega_0^2\left(\frac{l}{m}\right)^2\right] = R\,\Delta u - 4\,M\,\omega_0 r_0 \frac{l}{m}\,\Delta\omega\,,$$

$$F_1 = \frac{R}{R - 2\,M\,\omega_0^2\,(l/m)^2} = \frac{R}{R - B}\,, \quad F_2 = \frac{4\,M\,\omega_0 r_0\,l/m}{R - 2\,M\,\omega_0^2\,(l/m)^2} = \frac{A}{R - B}$$

The condition of static stability is

$$R > 2M\omega_0^2\left(\frac{l}{m}\right)^2$$

Numerical calculation

$$B = 2\,M\,\omega_0^2\,(l/m)^2 = 40 \cdot \frac{10^{-3}}{981} \cdot (300)^2 \cdot \left(\frac{1}{0{\cdot}7}\right)^2 = 7{\cdot}49 \text{ kg/cm}$$

The condition of static stability is satisfied.

$$A = 4\,M\,\omega_0 r_0\,l/m = 80 \cdot \frac{10^{-3}}{981} \cdot 300 \cdot 1{\cdot}2 \cdot \left(\frac{1}{0{\cdot}7}\right) = 4{\cdot}18 \cdot 10^{-2} \text{ kg/s}$$

$$F_1 = 2{\cdot}66 \text{ and } F_2 = 0{\cdot}930 \times 10^{-2}\frac{\text{cm}}{\text{rad/sec}}$$

(*ii*) *Estimation of F_4*

$$q = S\,dz/dl$$

$$\frac{\Delta z}{\Delta q} = \frac{1}{p\,S} = \frac{1}{3{\cdot}14\,p}\;\frac{\text{cm}}{\text{cm}^3/\text{s}}$$

(*iii*) *Estimation of F_5*

$$\frac{\Delta Q}{\Delta z} = \frac{\Delta Q}{\Delta S_R}\,\frac{\Delta S_R}{\Delta z}$$

Now since $\Delta P = \xi(\rho/2)(Q^2/S_R^2)$, we have

$$\frac{\Delta Q}{\Delta S_R} = \sqrt{\frac{2\,\Delta P}{\xi\,\varrho}}$$

and $\Delta S_R/\Delta z = k$, so that

$$\frac{\Delta Q}{\Delta z} = k\sqrt{\frac{2\,\Delta P}{\xi\,\varrho}}$$

Numerical calculation

$$\Delta P = 20 \text{ kg/cm}^2, \ \rho = \frac{0{\cdot}8 \times 10^{-3}}{981} \text{ g/cm}^3$$

$$\frac{\Delta Q}{\Delta S_R} = 5{,}300 \text{ cm/sec}$$

$$\frac{\Delta Q}{\Delta z} = 28 \frac{\text{cm}^3/\text{sec}}{\text{cm}}$$

(*iv*) *Estimation of F_6*

F_6 is effectively a first-order transfer function. The speed of the engine and the flow of fuel are related by the good approximation

$$\Delta Q = K \, \Delta C = K \left(\lambda \, \Delta \Omega + I \, \frac{\mathrm{d}\Omega}{\mathrm{d}t} \right)$$

where C is the torque absorbed and I the moment of inertia of the rotor.

We know that the speed lag is equal to the time constant. This gives $\tau = 0{\cdot}5$ sec.

By drawing a straight line through B parallel with the asymptote of the curve $N = f(t)$, we find, for example for $\Delta t = 1{\cdot}6$ sec

$$\Delta Q = 2 \text{ cm}^3/\text{sec and } \Delta\Omega = 915 \text{ rev/min}$$

wherefrom

$$\frac{\Delta\Omega}{\Delta Q} = \frac{457}{1+0{\cdot}5p} \frac{\text{rev/min}}{\text{cm}^3/\text{sec}} = \frac{48}{1+0{\cdot}5p} \frac{\text{rad/sec}}{\text{cm}^3/\text{sec}}$$

(*v*) *Open-loop transfer function of the regulator*

$(TF)_0 =$	F_2 ×	F_3 ×	F_4 ×	F_5 ×	F_6 ×	F_7
	$0{\cdot}930 \times 10^{-2}$	x	$\dfrac{1}{3{\cdot}14\,p}$	28	$\dfrac{48}{1+0{\cdot}5\,p}$	$\dfrac{1}{10}$
	$\dfrac{\text{cm}}{\text{rad/sec}}$	$\dfrac{\text{cm}^3/\text{sec}}{\text{cm}}$	$\dfrac{\text{cm}}{\text{cm}^3/\text{sec}}$	$\dfrac{\text{cm}^3/\text{sec}}{\text{cm}}$	$\dfrac{\text{rad/sec}}{\text{cm}^3/\text{sec}}$	no dimensions
	$\dfrac{\Delta x_2}{\Delta\omega}$	$\dfrac{\Delta q}{\Delta x}$	$\dfrac{\Delta z}{\Delta q}$	$\dfrac{\Delta Q}{\Delta z}$	$\dfrac{\Delta\Omega}{\Delta Q}$	$\dfrac{\Delta\omega}{\Delta\Omega}$

Therefore

$$(TF)_0 = \frac{0{\cdot}4\,x}{p\,(1 + 0{\cdot}5\,p)} = \frac{0{\cdot}2\,x}{0{\cdot}5\,p\,(1 + 0{\cdot}5\,p)}$$

Draw the curve of $1/0{\cdot}5p(1+0{\cdot}5p)$ on the Nichols chart.

If there is to be no amplitude rise in the closed loop, we must lower the curve by 5 dB. The required transfer function is therefore

$$\frac{0{\cdot}56}{0{\cdot}5p(1+0{\cdot}5p)}$$

so that

$$x = \frac{0{\cdot}56}{0{\cdot}2} = 2{\cdot}8$$

To attain such a small value technologically is almost impossible. If it were possible, the resulting regulator would be much too slow. A compensating network is necessary.

II. Regulator with hydraulic compensating network

(*i*) (1) $\Delta q_1 = C(\Delta P_m - \Delta P_v)$ (flow through the orifice O);
(2) $\Delta q_2 = \Sigma p \Delta y$ (displacement of piston B);
(3) $K\Delta y = \Sigma(\Delta P_m - \Delta P_v)$ (force equation for piston B, assumed to have no inertia)
(4) $\Delta q_1 + \Delta q_2 = \Delta q$ (conservation of flow)

Eliminating Δy between (2) and (3) gives

$$\Delta q_2 = \frac{\Sigma^2}{K} p\,(\Delta P_m - \Delta P_v) \tag{2'}$$

and substituting eqn. (1) and (2′) in (4)

$$(\Delta P_m - \Delta P_v)\left(C + \frac{\Sigma^2}{K} p\right) = \Delta q$$

Since $\Delta q = F_3 \Delta x$

$$\Delta P_m - \Delta P_v = \Delta x \frac{F_3/C}{1 + (\Sigma^2/KC)\cdot p}$$

(*ii*) The equation of equilibrium of the centrifugal weights and valve-spool assembly is

$$\Delta F_M + \Delta F_R + \Delta F_C = 0$$

It has been shown that:

$$\Delta F_M = -A\,\Delta\omega + B\Delta x$$

$$\Delta F_R = (\Delta u - \Delta x)\,R$$

Calculation of ΔF_c

$$\Delta F_C = -(\Delta P_m - \Delta P_v)\,\sigma = -\Delta x \frac{F_3\,\sigma/C}{1 + (\Sigma^2/KC)\cdot p}$$

wherefrom

$$-A\,\Delta\omega + B\,\Delta x + R\,\Delta u - R\,\Delta x - \Delta x\,\frac{F_3\,\sigma/C}{1 + (\Sigma^2/KC)\cdot p} = 0$$

$$\Delta x\left[R - B + \frac{F_3\,\sigma/C}{1 + (\Sigma^2/KC)\cdot p}\right] = R\,\Delta u - A\,\Delta\omega$$

The two transfer functions F_1 and F_2 must be multiplied by

$$A_1 = A_2 = \frac{1}{1 + \dfrac{F_3\,\sigma}{C\,(R-B)}}\cdot\frac{1 + (\Sigma^2/KC)\cdot p}{1 + \dfrac{\Sigma^2/KC}{1 + \dfrac{F_3\,\sigma}{C\,(R-B)}}\cdot p}$$

(*iii*) The normal form of a phase-advancing compensator is

$$\frac{1}{a}\,\frac{1 + a\,\tau\,p}{1 + \tau\,p}$$

so that

$$a = 1 + \frac{F_3\,\sigma}{C\,(R-B)}$$

and

$$a\,\tau = \frac{\Sigma^2}{KC}$$

Numerical calculation

$$\alpha\tau = 0{\cdot}5 \text{ sec}$$

$$\alpha = 1 + \frac{1{,}150\times 0{\cdot}8}{2\times 4{\cdot}65} = 100$$

$$\therefore\ \tau = 0{\cdot}005 \text{ sec}$$

The transfer function of the compensating network is

$$\frac{1}{100}\,\frac{1 + 0{\cdot}5\,p}{1 + 0{\cdot}005\,p}$$

The overall transfer function is:

$$\frac{0{\cdot}2\times 1{,}150}{0{\cdot}5p(1+0{\cdot}5p)}\,\frac{1}{100}\,\frac{(1+0{\cdot}5p)}{(1+0{\cdot}005p)} = \frac{2{\cdot}3\times 10^{-2}}{0{\cdot}005p(1+0{\cdot}005p)}\text{: stable.}$$

The transfer function without the compensating network is

$$\frac{2{\cdot}3\times 10^{-2}}{0{\cdot}5p(1+0{\cdot}5p)}\text{: unstable}$$

The stability margin appears to be ample. In practice, it is less (due to simplifications made in the analysis). On the other hand, the regulator must be stable at all flight conditions, and the high-altitude conditions are much more severe.

EXAMPLE 9

(*i*) *The initial force*, F_0, in the flapper-plate springs balances the force due to the pressure $P_2\sigma_2$:

$$P_2 = 37 \cdot 5 \text{ kg/cm}^2$$

$$\sigma_2 = 0 \cdot 385 \times 10^{-2} \text{ cm}^2$$

$$F_0 = P_2\sigma_2 = 0 \cdot 145 \text{ kg}$$

The *flow*, q_0, is found from the flow equation of the fixed orifice

$$P_1 - P_2 = \xi_1 \frac{\varrho^*}{2} \left(\frac{q_0}{s_1}\right)^2$$

wherefrom, by using e.g. Graph D, $q_0 = 2 \cdot 2$ cm³/sec. Thus, for the two orifices, the flow is $4 \cdot 4$ cm³/sec.

The *distance*, x_0, is found from the flow equation of the jet

$$P_2 = \xi_2 \frac{\varrho^*}{2} \left(\frac{q_0}{s_{20}}\right)^2$$

from which, again using Graph D

$$s_{2_0} = 2 \cdot 6 \times 10^{-4} \text{ cm}^2 \text{ and } X_0 = 1 \cdot 18 \times 10^{-3} \text{ cm } (12\ \mu)$$

(*ii*) *Calculation of the gain K*

Electrical torque	$K_1 i$
Opposing hydraulic torque . .	$2P\sigma_2 L/2 = P\sigma_2 L$
Hydraulic force on the valve . .	$2PS$
Opposing force from the springs .	Ry

By eliminating P, we have

$$K = (y/i)_0 = \frac{2\,K_1\,S}{L\,R\,\sigma_2} = 6 \cdot 12 \text{ cm/A}$$

For $i = 12$ mA, $y = 73 \cdot 3 \times 10^{-3}$ cm $= 0 \cdot 733$ mm

(*iii*) *The functional equations*

The torque equation for the flapper plate is

$$K_1 i - PL\sigma_2 = \theta\,(\varrho + \varphi p + jp^2) \tag{1}$$

and that of the forces acting on the valve

$$2\,SP = y\,(R + Fp + Mp^2) \tag{2}$$

The equation of the flow through the hydraulic potentiometer (equation of the variations) is

$$q_e - q_s = S\,p\,y + \frac{V}{B}\,p\,P$$

Now, $q_e = ks_1\sqrt{P_1 - P''}$ gives, after differentiation

$$\frac{\mathrm{d}q_e}{q_0} = \frac{1}{2}\,\frac{\mathrm{d}\,(P_1 - P'')}{(P_1 - P'')_0} = -\frac{1}{2}\,\frac{P}{P_1 - P_2}$$

and similarly, $q_s = kS_2\sqrt{P''}$ gives

$$\frac{\mathrm{d}q_s}{q_0} = \frac{\mathrm{d}s_2}{s_{20}} + \frac{1}{2}\,\frac{\mathrm{d}P''}{P''_0} = -\frac{x}{X_0} + \frac{1}{2}\,\frac{P}{P_2}$$

so that

$$\left(\frac{q_0 P_1}{2\,P_2\,(P_1 - P_2)} + \frac{V}{B}\,p\right) P + \frac{q_0}{X_0}\,x + S\,p\,y = 0$$

Replacing x by $\theta L/2$, $q_0P_1/2P_2(P_1 - P_2)$ by α and $Lq_0/2X_0$ by β, this becomes

$$\left(\alpha + \frac{V}{B}\,p\right) P - \beta\,\theta + S\,p\,y = 0 \tag{3}$$

By writing the three equations obtained in the form

$$\theta = \frac{1}{\varrho + \varphi p + j p^2} \cdot [K_1 i - L\,\sigma_2 P] \tag{1'}$$

$$P = \frac{1}{\alpha + \dfrac{V}{B}\,p} \cdot [\beta\,\theta - S\,p\,y] \tag{2'}$$

$$y = \frac{1}{R + F p + M p^2} \cdot 2\,S P \tag{3'}$$

the form of the block diagram (*Figure 10.20*) is obvious.

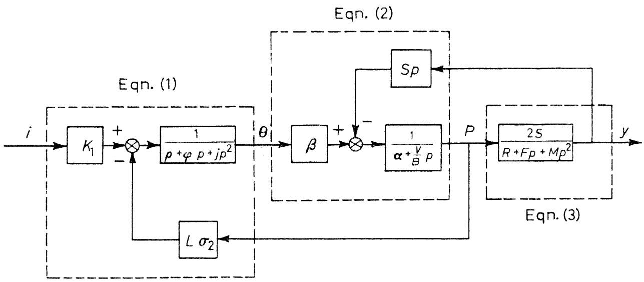


Figure 10.20

The use of an imaginary open chain is not very helpful. It is preferable to examine the effect of the servo valve on the overall transfer function y/i. Elimination of θ and P between the three functional equations gives the transfer function

$$\frac{y}{i} = \frac{N}{D_0 + D_1 p + D_2 p^2 + D_3 p^3 + D_4 p^4 + D_5 p^5}$$

where

$$D_0 = R\left[\underline{\underline{\beta L \sigma_2}} + \alpha \varrho\right] \dots\dots\dots\dots\dots\dots$$

$$D_1 = F\left[\underline{\underline{\beta L \sigma_2}} + \alpha \varrho\right] + R\left[\frac{V}{B}\varrho + \alpha\varphi\right] \dots\dots\dots + \underline{2\,S^2\,\varrho}$$

$$D_2 = M\left[\beta L \sigma_2 + \alpha \varrho\right] + F\left[\frac{V}{\underline{B}}\varrho + \alpha\varphi\right] + R\left[\frac{V}{B}\varphi + \alpha j\right] \dots + \underline{\underline{2\,S^2\,\varphi}}$$

$$D_3 = \dots\dots\dots M\left[\frac{V}{B}\varrho + \alpha\varphi\right] + F\left[\frac{V}{\underline{B}}\varphi + \alpha j\right] + R\frac{V}{B}j + \underline{2\,S^2\,j}$$

$$D_4 = \dots\dots\dots\dots\dots\dots M\left[\frac{V}{B}\varphi + \alpha j\right] + \underline{F\frac{V}{B}j} \dots$$

$$D_5 = \dots\dots\dots\dots\dots\dots\dots\dots \underline{M\frac{V}{B}j} \dots$$

$$N = \underline{\underline{2\,S\,\beta\,K_1}} \dots\dots\dots\dots\dots\dots$$

Numerical calculation

$$\alpha = 3{\cdot}91 \times 10^{-2} \qquad \beta = 2{\cdot}33 \times 10^{3} \qquad N = 20{\cdot}6 \times 10^{3}$$

$$\left.\begin{aligned} \beta L\sigma_2 &= 22{\cdot}4 \\ \alpha\rho &= 1{\cdot}2 \end{aligned}\right\} \beta L\sigma_2 + \alpha\rho = 23{\cdot}6$$

$\alpha\rho$ is small compared with $\beta L\sigma_2$ and can be neglected in a first approximation.

$$\left.\begin{aligned} \frac{V}{B}\varrho &= 20 \cdot 10^{-4} \\ \alpha\varphi &= 3{\cdot}9 \cdot 10^{-4} \end{aligned}\right\} \frac{V}{B}\varrho + \alpha\varphi = 23{\cdot}9 \cdot 10^{-4}$$

Strictly, $\alpha\varphi$ could be neglected in comparison with $(V/B)\rho$.

$$\left.\begin{aligned} \frac{V}{B}\varphi &= 67 \cdot 10^{-8} \\ \alpha j &= 7{\cdot}8 \cdot 10^{-8} \end{aligned}\right\} \frac{V}{B}\varphi + \alpha j = 74{\cdot}8 \cdot 10^{-8}$$

αj can be neglected compared with $(V/\beta)\varphi$.

$$\frac{V}{B}j = 1{\cdot}34 \cdot 10^{-10}$$

$$2\,S^2 = 0{\cdot}0626$$

The exact value of the gain is

$$K = N/D_0 = 5{\cdot}82$$

compared with that of 6·12 obtained by neglecting $\alpha\rho$ as against $\beta L\sigma_2$ in D_0.

Using the same layout as when using the symbols, the numerical results are

$10^{-3}\cdot D_0 =$	$\underline{\underline{3\cdot54}}$					$= 3\cdot54$
$D_1 =$	$\underline{\underline{4\cdot7}}$	$+\ 0\cdot36$			$+\ \underline{1\cdot88}$	$= 6\cdot94$
$10^3\cdot D_2 =$	$0\cdot12$	$+\ \underline{\underline{0\cdot48}}$	$+\ 0\cdot11$		$+\ \underline{\underline{0\cdot62}}$	$= 1\cdot33$
$10^6\cdot D_3 =$		$0\cdot012$	$+\ \underline{0\cdot15}$	$+\ 0\cdot02$	$+\ \underline{0\cdot125}$	$= 0\cdot3$
$10^9\cdot D_4 =$			$0\cdot004$	$+\ \underline{0\cdot027}$		$= 0\cdot03$
$10^{12}\cdot D_5 =$				$\underline{0\cdot0007}$		$= 0\cdot0007$
$10^{-3}\cdot N$						$= 20\cdot6$

We can now plot the frequency response curve* and examine possible resonances in the system.

The table also enables us to assess the effects of the different parameters on the performance of the servo valve. For example, it can be seen that up to $\omega = 10^3$, the transfer function can be considered as third-order or even second-order (with $\omega_0 = 1{,}630$ and $\xi = 1\cdot6$). It can also be seen that, at low frequencies, the servo valve can be represented by its gain and the time constant $\tau = D_1/D_0 = 2\times10^{-3}$ sec.

To show the effect of the different parameters, the most important terms have been underlined in the above tables for the transfer function.

(*iv*) If C_e is the electrical torque, the two servo valves have the transfer functions

$$\frac{Q}{C_e} = \frac{N K_2/K_1}{D_0 + D_1 p + D_2 p^2 + \cdots}$$

and

$$\frac{Q}{C_e - C_Q} = \frac{N K_2/K_1}{D'_1 p + D'_2 p^2 + \cdots} \qquad (\text{i.e. } D'_0 = 0)$$

Therefore, for the closed loop of the second

$$\frac{Q}{C_e} = \frac{\dfrac{N K_2/K_1}{D'_1 p + D'_2 p^2 + \cdots}}{1 + \dfrac{H_0 N K_2/K_1}{D'_1 p + D'_2 p^2 + \cdots}} = \frac{N K_2/K_1}{H_0 N K_2/K_1 + D'_1 p + \cdots}$$

To maintain the same value of the gain, we require that

$$H_0 N K_2/K_1 = D_0,$$

$$H_0 = \frac{D_0 K_1}{N K_2} = \frac{K_1}{K K_2} = \frac{4\cdot3}{K_2}$$

*Or calculate the roots of the denominator of the transfer function.

The numerical table shows that, in practice, R has little effect, except in D_0, so that $D_1' \simeq D_1 D_2' \simeq D_2$, etc. The dynamic performance is effectively conserved.

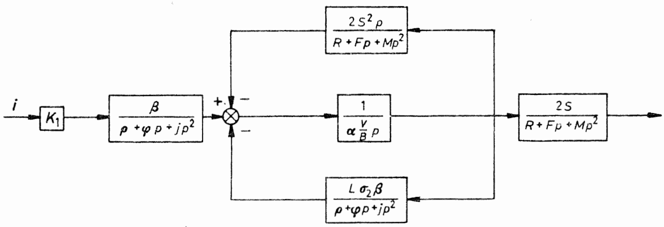


Figure 10.21

Note—The block diagram may be put in the form shown in *Figure 10.21*, in which case the stability analysis reduces to a consideration of the open-loop transfer function

$$\frac{1}{\left(\alpha+\dfrac{V}{B}p\right)}\left[\frac{2S^2p(\rho+\phi p+jp^2)+L\sigma_2\beta(R+Fp+Mp^2)}{(R+Fp+Mp^2)(\rho+\phi p+jp^2)}\right]$$

However, these considerations are no longer important, since analogue computers can assess components with this type of transfer function very much more quickly and easily.

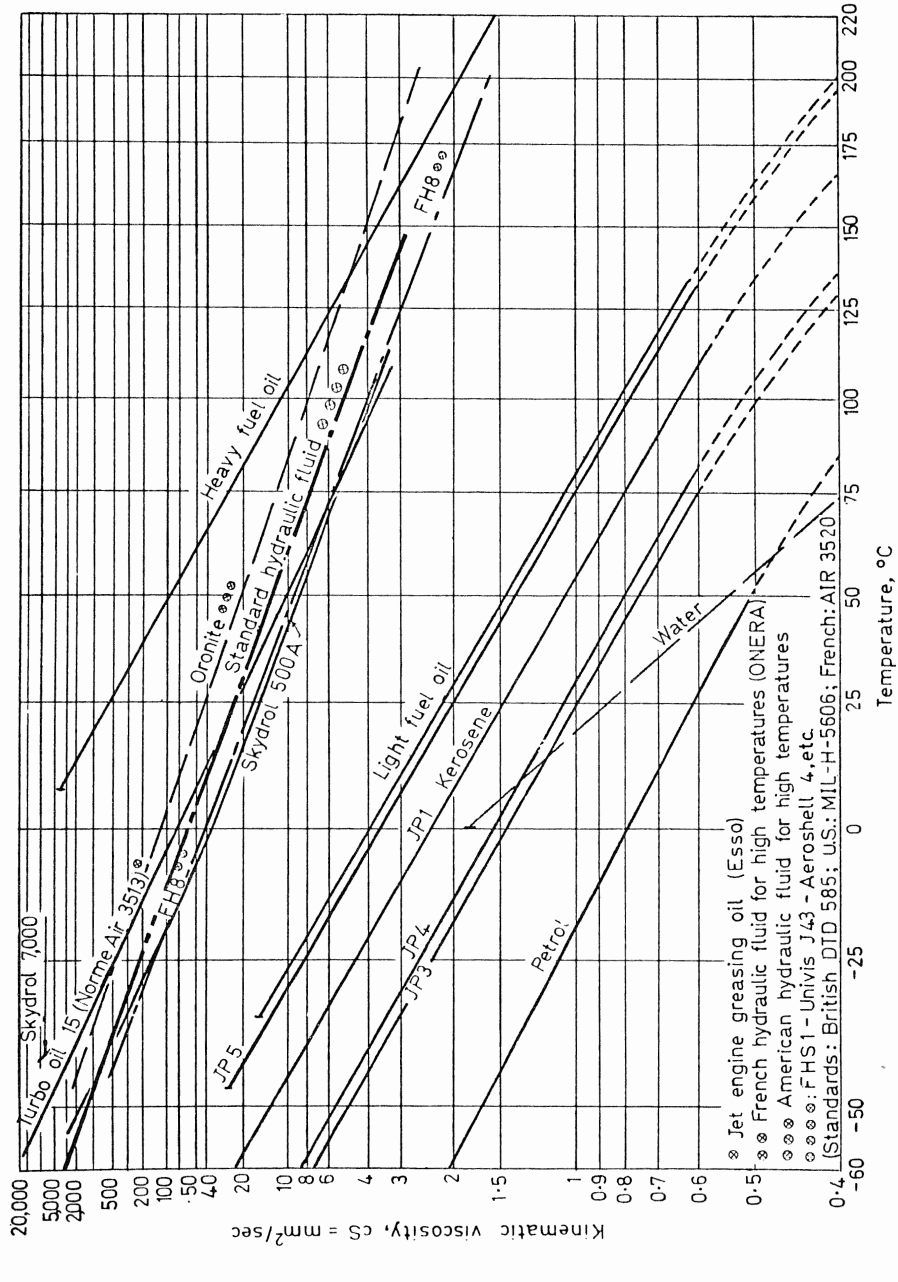
11

GRAPHS AND TABLES

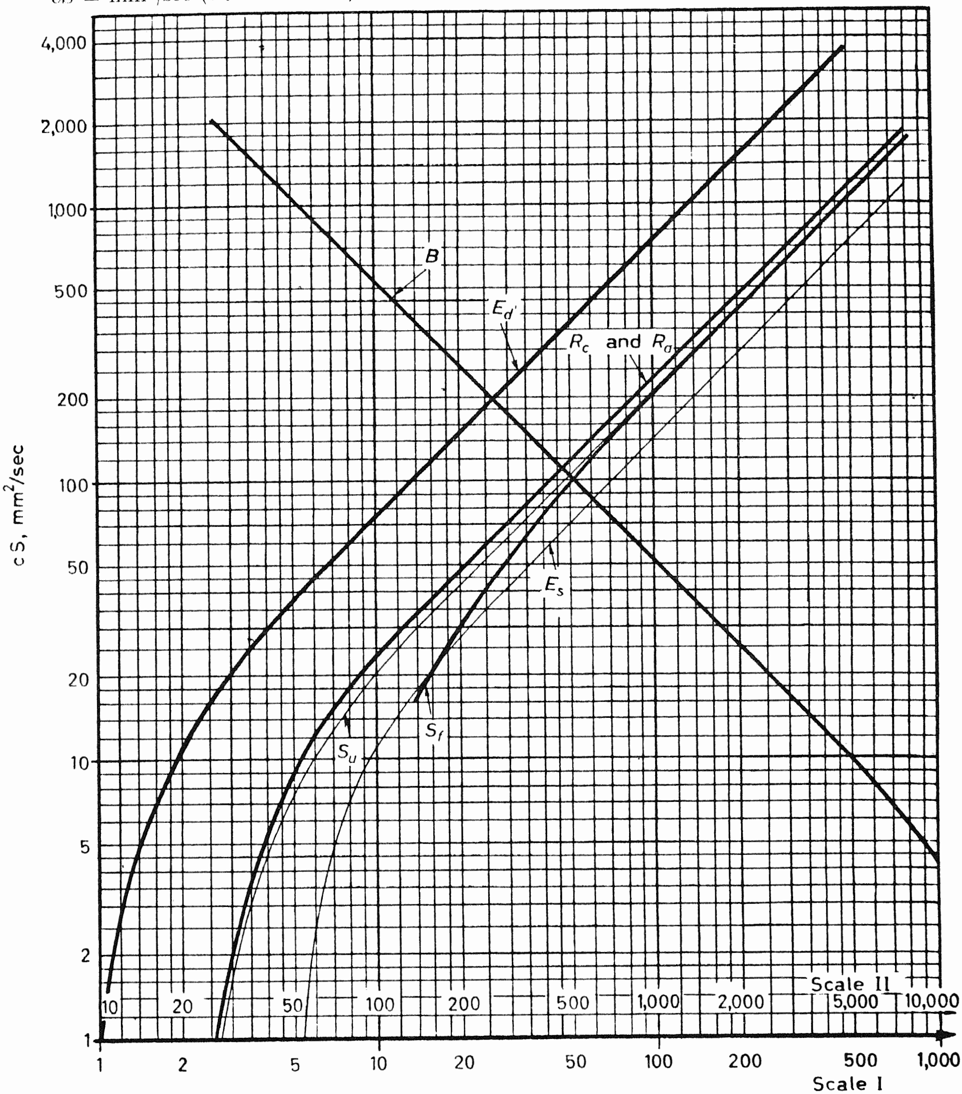
This Chapter consists of graphs giving information of general interest to the hydraulic engineer. The first two, with values of the viscosities of various fluids and conversion factors, are followed by others which will be useful for calculating the losses of head in hydraulic systems. Finally there are Hall's and Nichol's charts with various transfer loci, together with a conversion scale for decibels. A note on units, added at the end of the Chapter, will be of assistance to the reader in assessing the orders of magnitude of the continental units used in this book.

In addition to these, the following graphs, included in the main text, may well be of interest to the reader.

Graph A. Variation of the viscosity of different liquids with temperature (cf. Section 1.1)

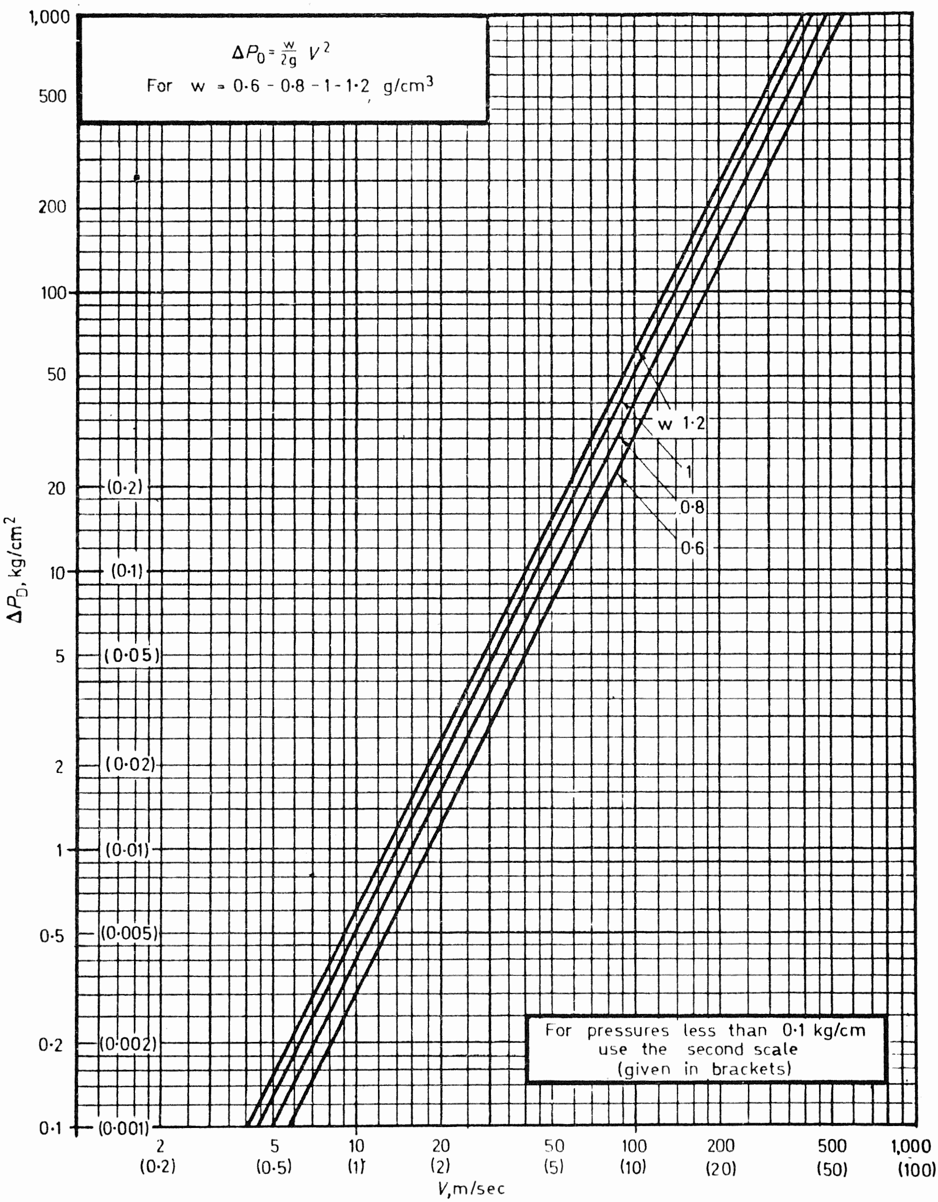


Graph B. Conversion of Engler, Barbey, Saybolt and Redwood units of viscosity to cS = mm²/sec (cf. Section 1.2)



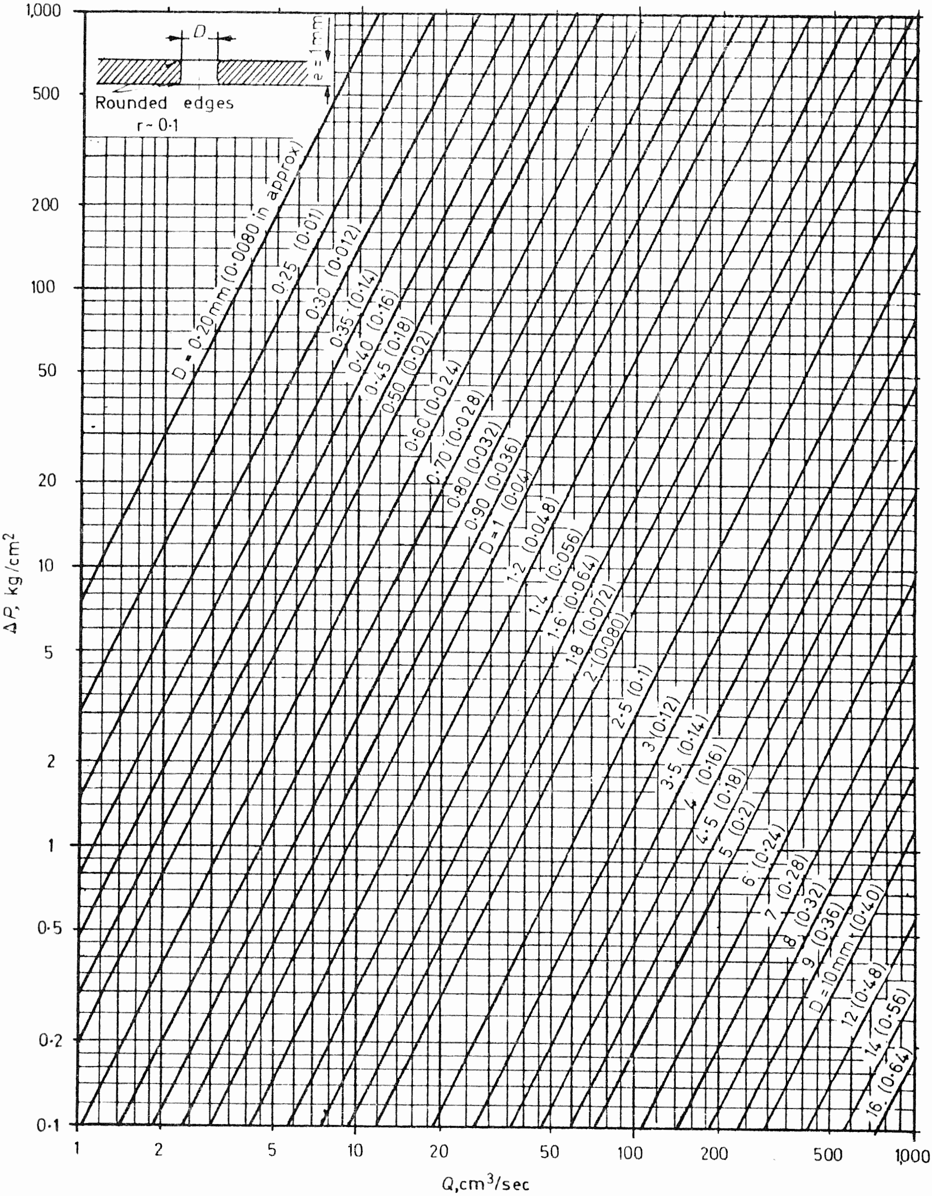
Units		*Curve*	*Read off scale*
Engler	degrees	E_d	I
	sec	E_s	II
Barbey		B	I
Saybolt	universal	S_u	II
	furol	S_f	I
Redwood	commercial	R_c	II
	admiralty	R_a	I

Graph C. Variation of dynamic pressure with velocity for different values of the specific weight, w (cf. Section 1.3)

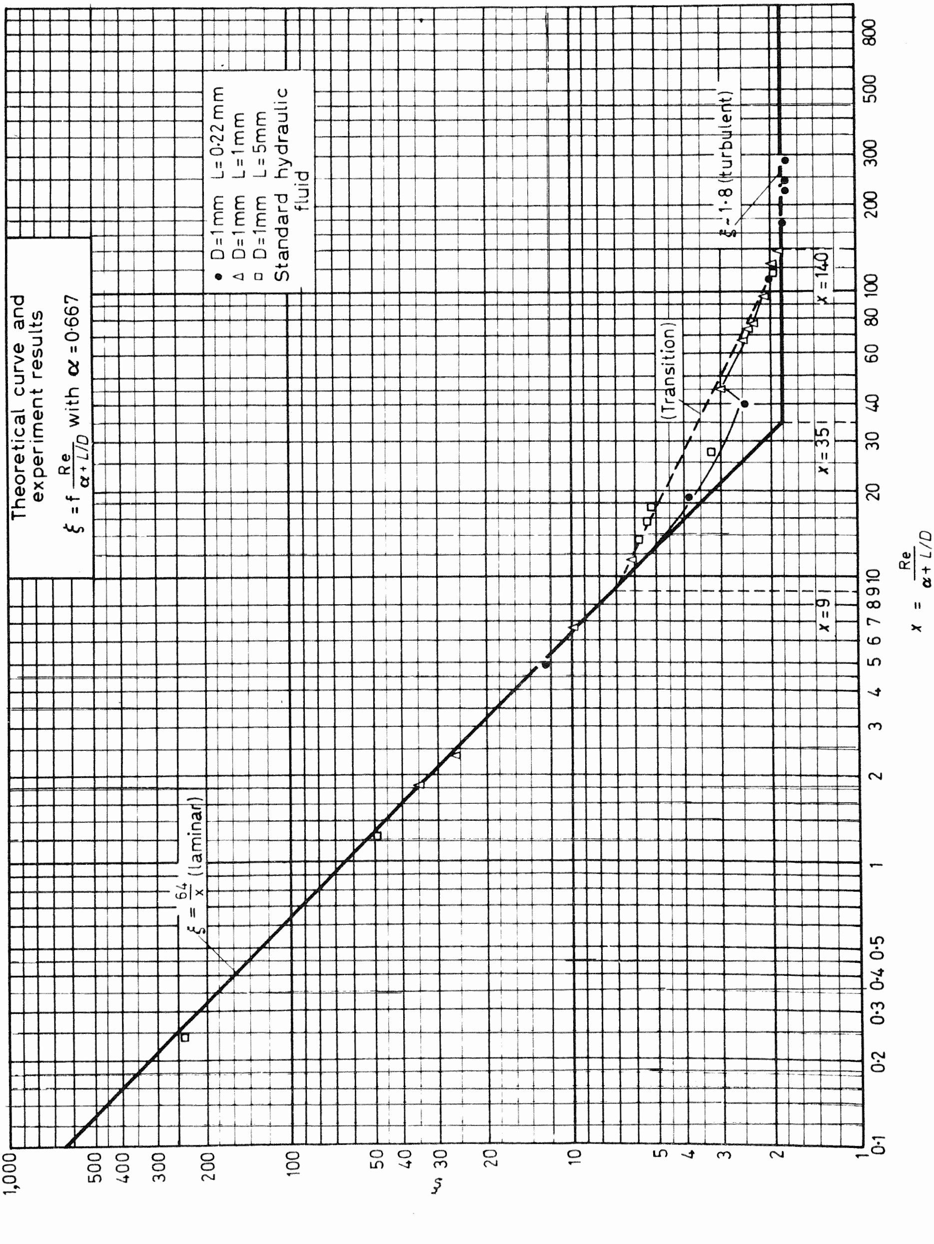


Graph D. Variation of loss of head for turbulent flow (cf. Section 1.4) through a circular orifice with flow Q for different values of the orifice diameter

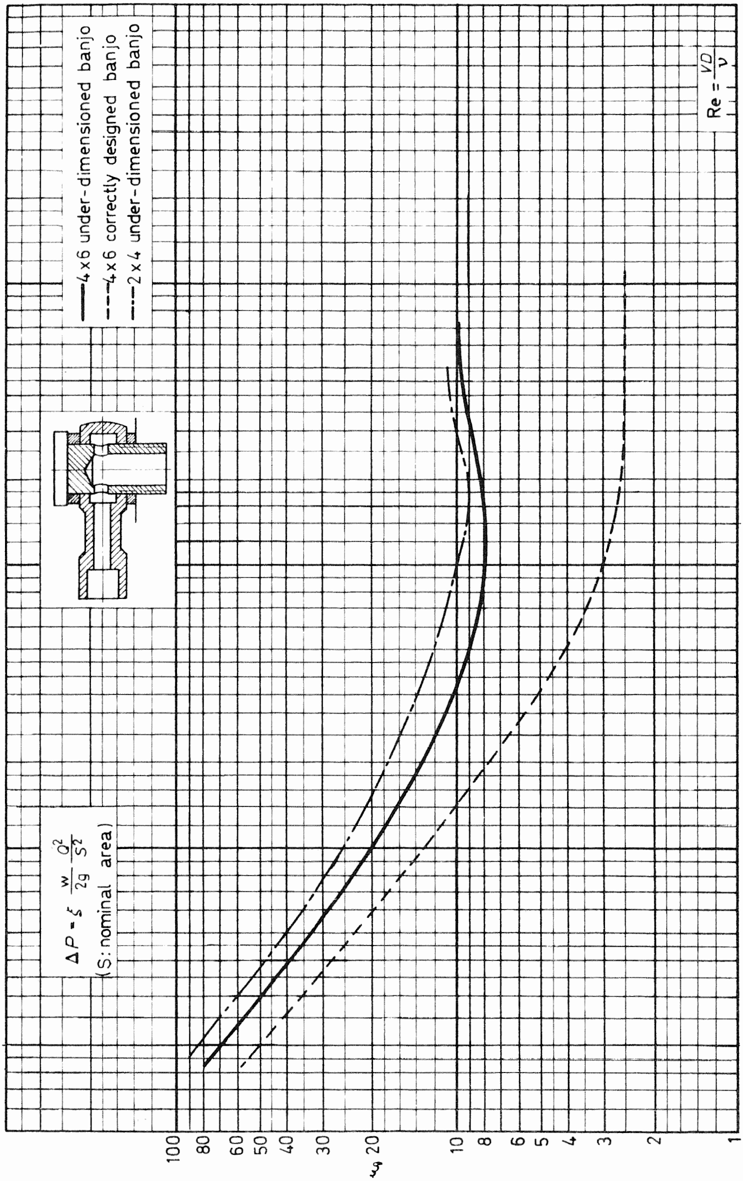
$$\Delta P = \frac{\xi w}{2g}\left[\frac{Q}{\pi/4D^2}\right]^2 \quad \text{where} \quad \begin{cases} w = 0{\cdot}8 \text{ g/cm}^3 \text{ (kerosene)} \\ \xi = 1{\cdot}8 \end{cases}$$



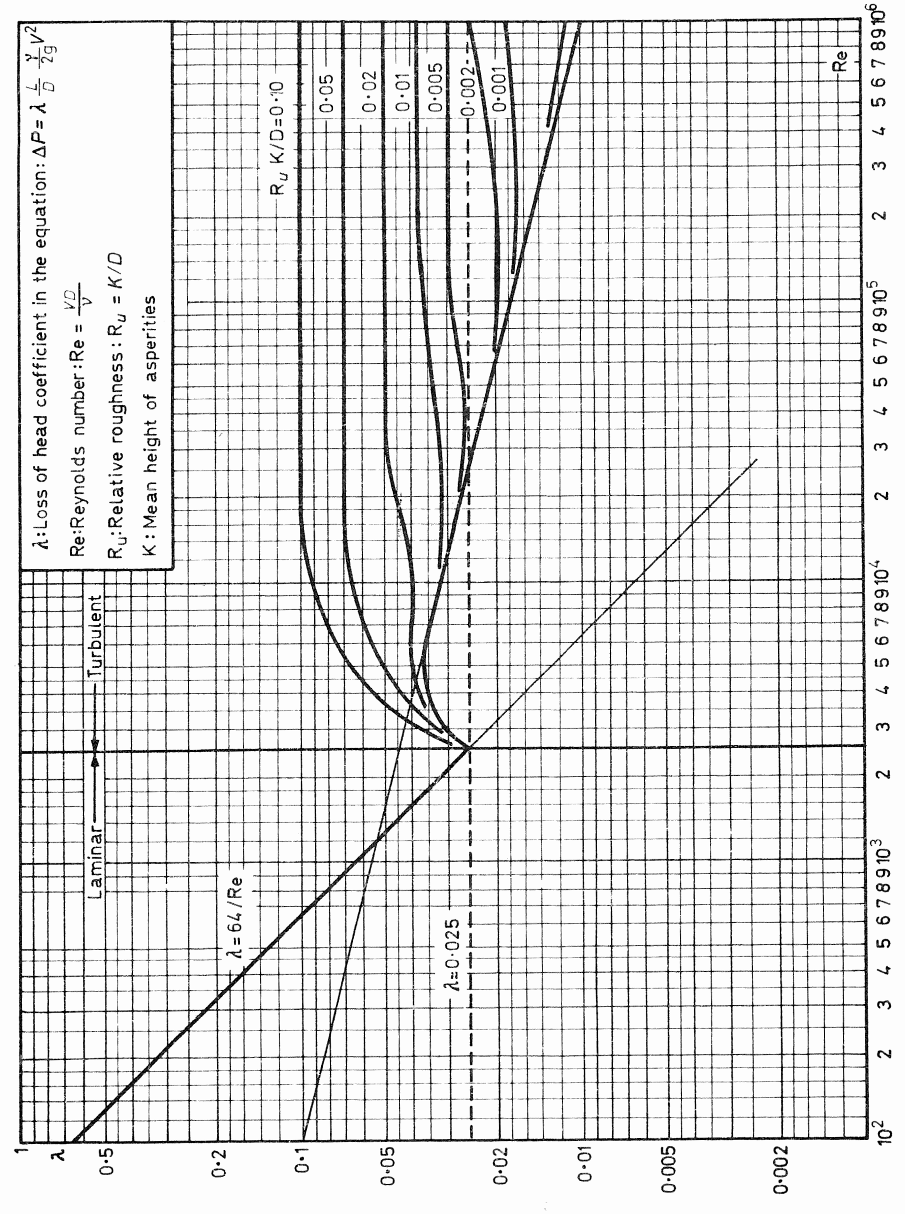
Graph E. Variation of head loss coefficient ξ with Reynolds number for a sharp-edged circular orifice, diameter D, length L (cf. Section 1.5)



Graph F. Variation of loss of head coefficient ξ with Reynolds number for some unions (experimental curves; cf. Section 1.6)



Graph G. Variation of the loss of head coefficient, λ, with Reynolds number for pipes of different roughness (cf. Section 1.7)

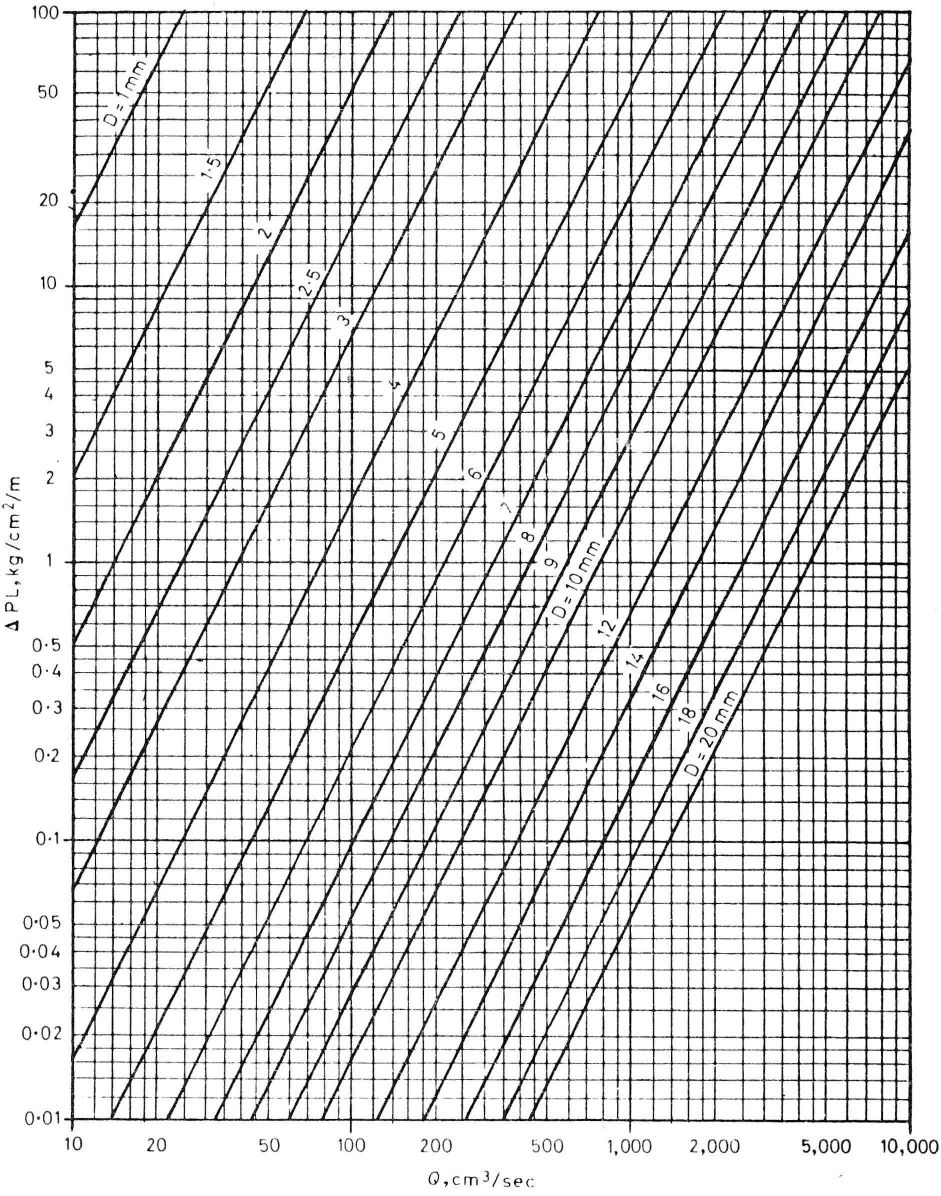


Graph H. Variation of the loss of head for turbulent flow in a pipe with volume flow for different diameters (cf. Section 1.8)

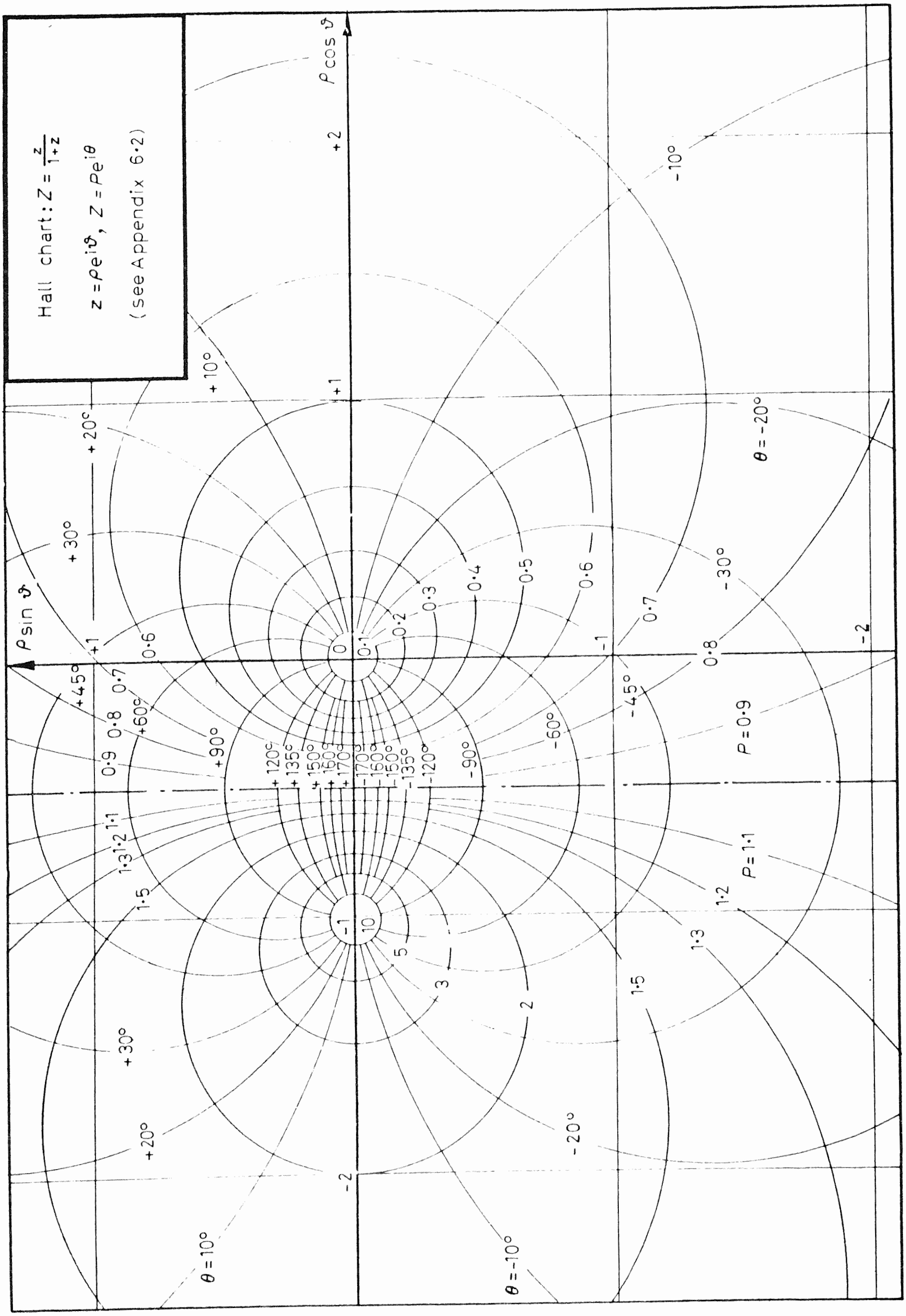
$$\frac{\Delta P}{L} = \lambda \frac{L}{D} \frac{w}{2g}\frac{Q^2}{S^2}, \text{ with } \lambda = 0{\cdot}025,\ w = 0{\cdot}8 \text{ g/cm}^3$$

$$\left(\text{with } \frac{\Delta P}{L}, \text{ kg/cm}^2 \text{ per m of tube, } Q, \text{ cm}^3\text{/sec, } D, \text{ mm: } \frac{\Delta P}{L} = 0{\cdot}165\left[\frac{Q^2}{D^5}\right]\right)$$

Note—For $w \neq 0{\cdot}8$, multiply $\frac{\Delta P}{L}$ by $w/0{\cdot}8$.



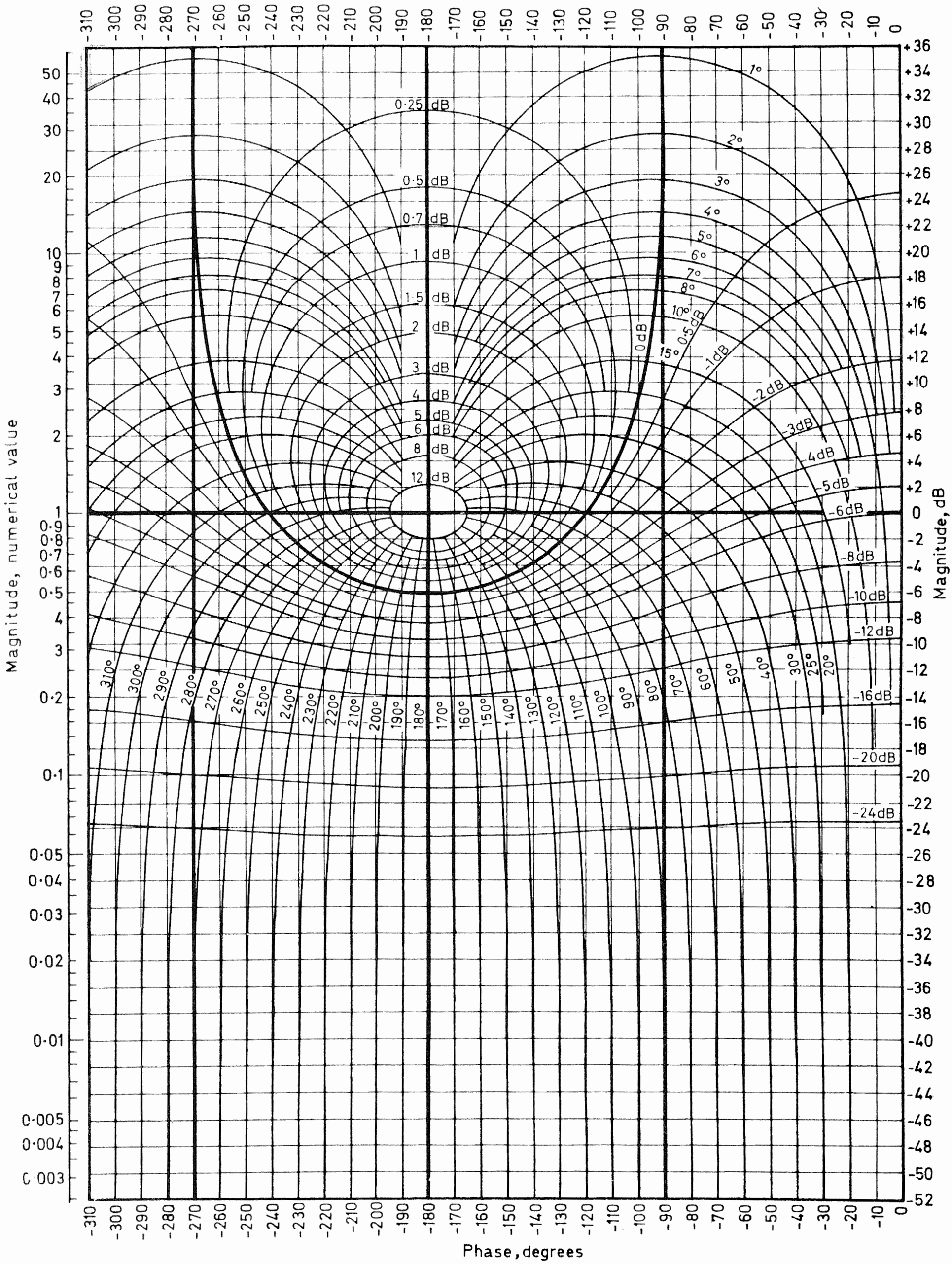
Graph I. Hall chart (cf. Section 6.1)



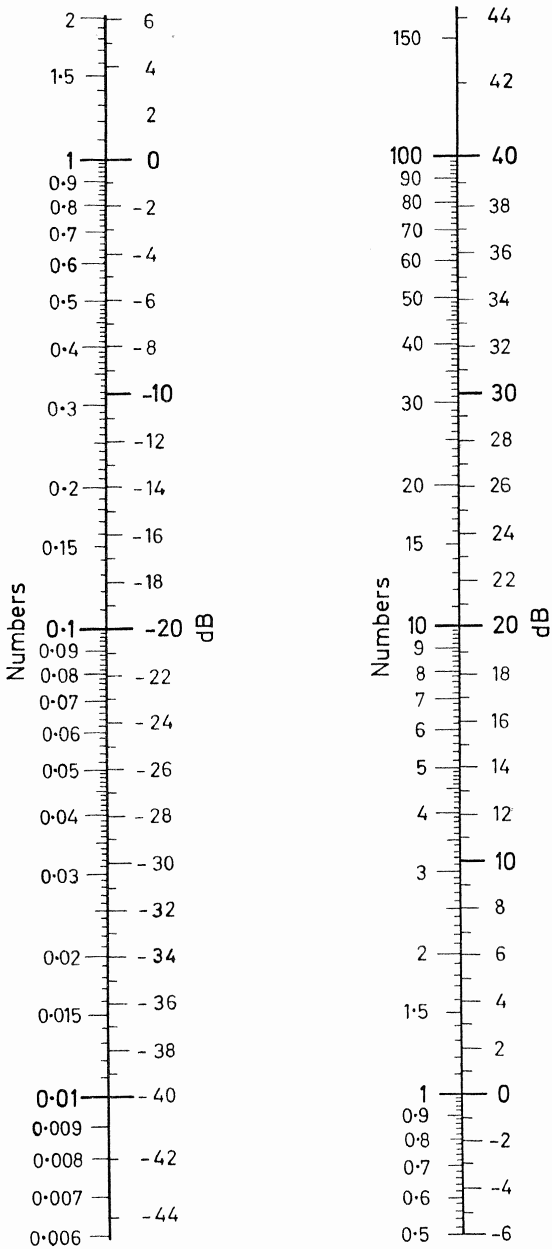
Graph J. Nichols chart: $Z = z/(1+z)$ (cf. Section 6.2)

Rectangular coordinates, z

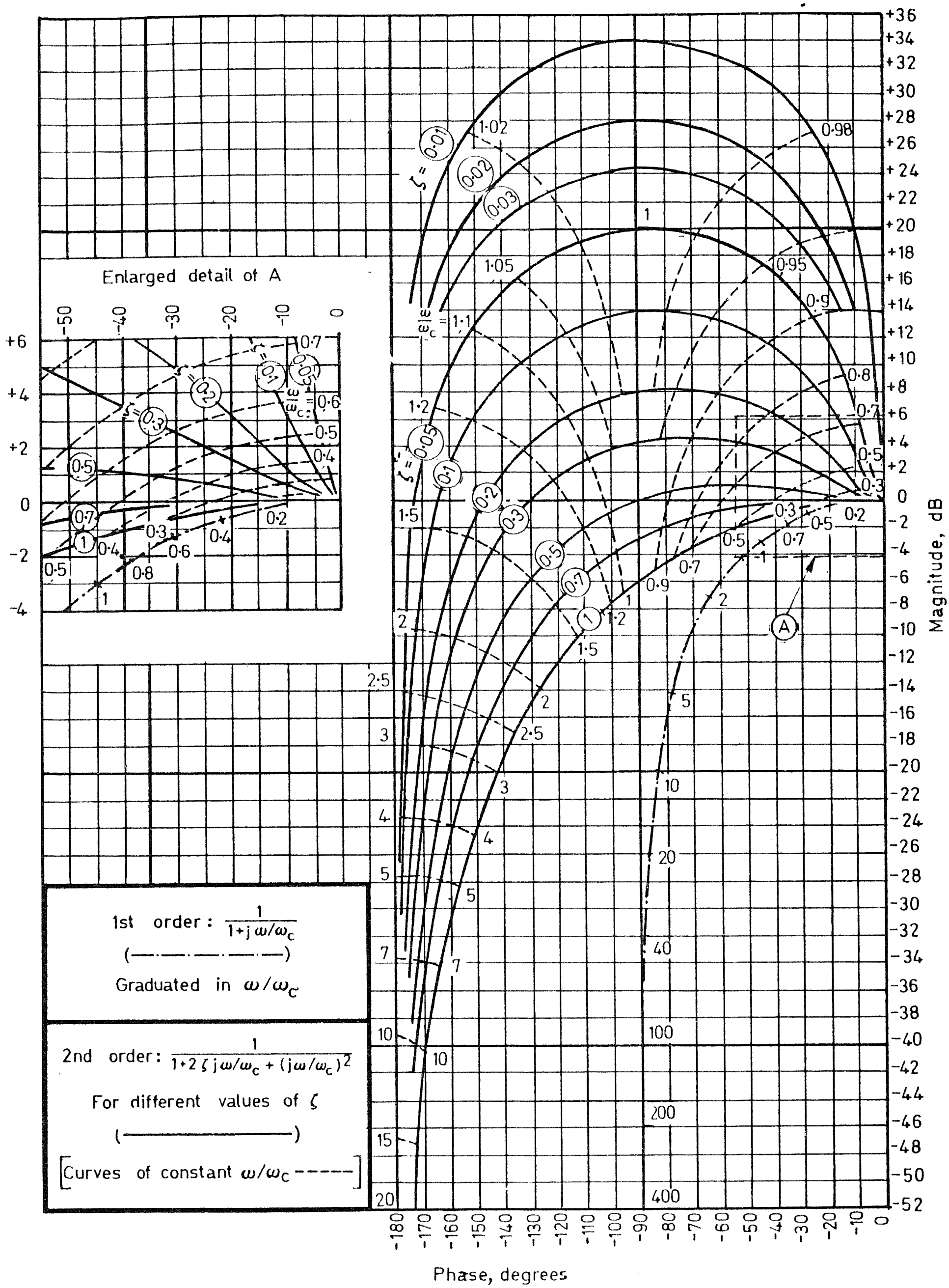
Curvilinear coordinates, Z



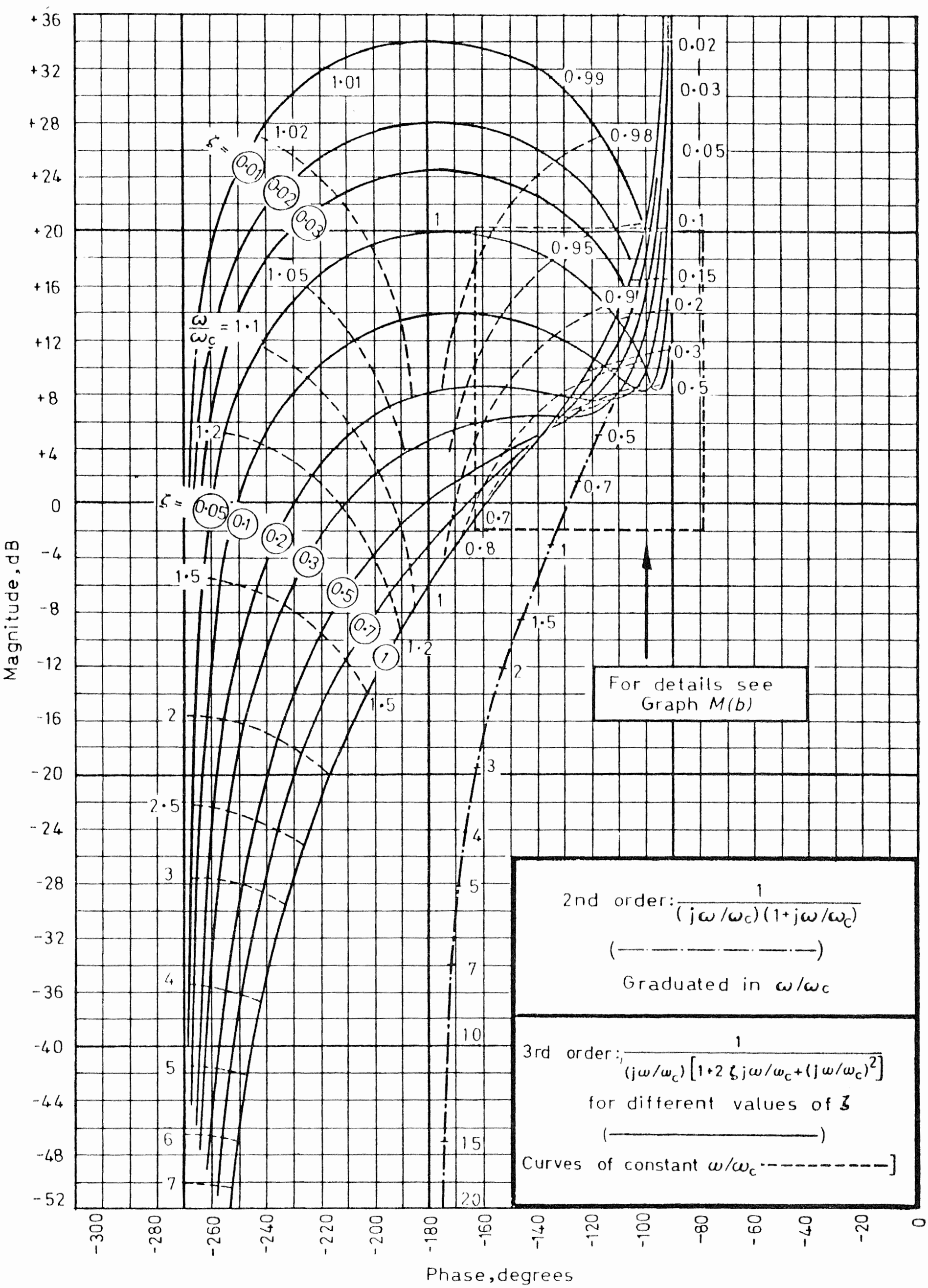
Graph K. Conversion to decibels (cf. Section 6.3)



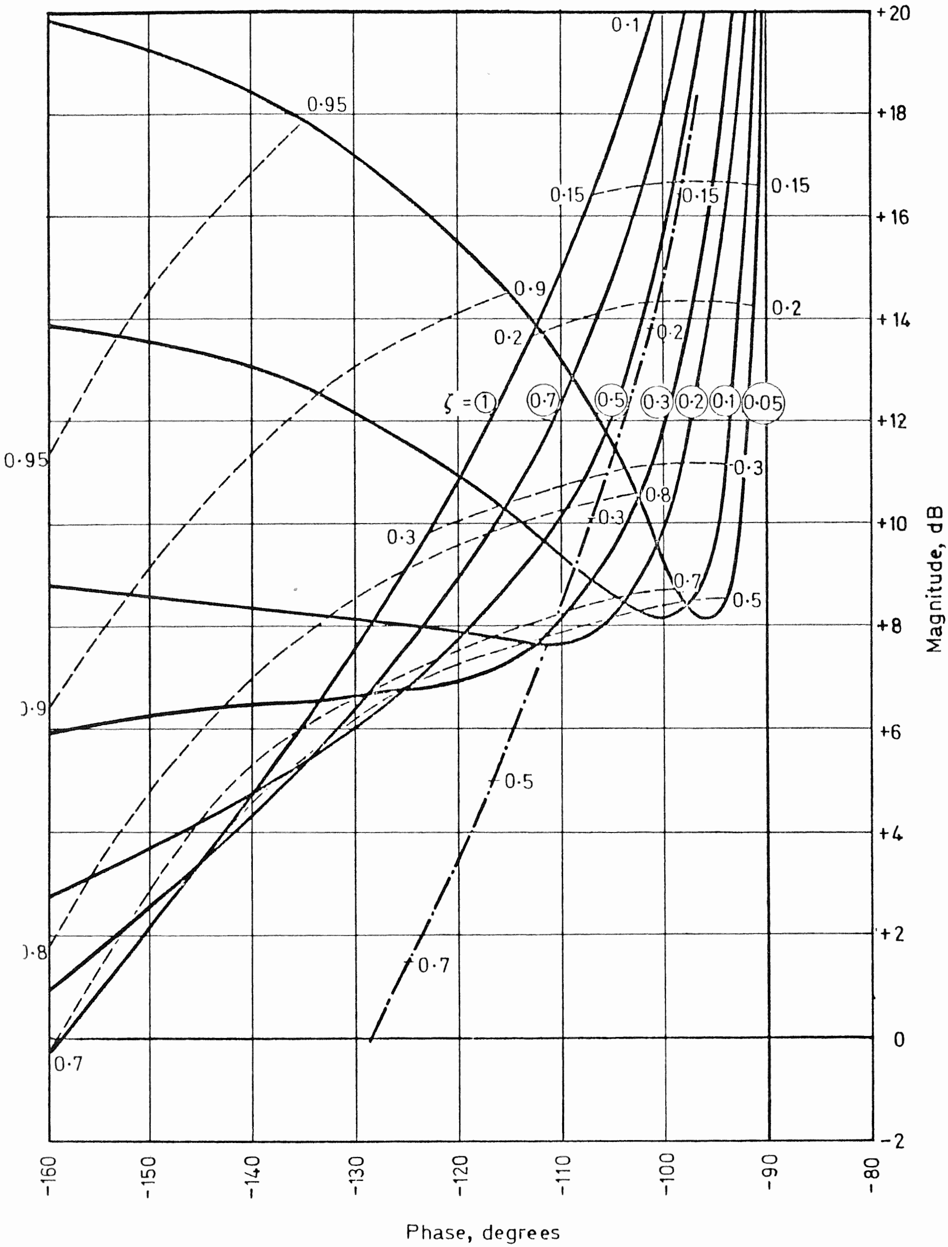
Graph L. First- and second-order transfer loci—(Nichols coordinates)



Graph M. (*a*) Second- and third-order transfer loci with integration (cf. Section 6.5)



Graph M. (*b*) scaled-up detail



UNITS

Translator's note—The numerical values in this book are, of course, quoted in metric units. Owing to the practical nature of the book and the commendable way in which the author illustrates the application of the analyses with suitable examples, it would be a major task to rewrite it using British units. Indeed, as the author says in his original note on units, 'whatever his opinion on the relative merits of different systems of units, the hydraulic engineer must be familiar with them all'.

However, in view of the controversy that exists over the 'slug' and the 'poundal' and the inability of some of our younger engineers at least even to differentiate between the pound mass and the pound weight, conversion tables have been included as a guide for the reader, to enable him to appreciate better the significance of the numerical examples.

Table U1 gives the conversion factors for the fundamental quantities of mass, force, length and temperature while *Tables U2 and U3* deal briefly with the more common units of some derived quantities, such as pressure, density and viscosity.

Table U1

Quantities	*French legal system* SI (MKSA)[1]	*Other metric systems* MKfS[2]	MTS[3]	CGS[4]	*Abbreviation*	*British equivalent*
Length	metre				m	3·28 ft 39·37 in
		metre			m	
			metre		m	
				centimetre[5]	cm	0·3937 in
Mass	kilogramme mass				kg	2·205 lb 0·0685 slug
		—			$\frac{\text{kgf.s2}}{\text{m}}$	21·62 lb 0·672 slug
			tonne mass[6]		t	
				gramme mass[6]	g	0·002205 lb 0·0353 oz
Force	newton[7]				N	7·233 pdl 0·225 lb weight
		kilogramme force[7]			kg (f)	70·93 pdl 2·205 lb weight
			sthene[7]		sn	
				dyne[7]	dyn	$7 \cdot 233 \times 10^{-5}$ pdl
Torque Energy Work done	newton-metre joule[8]				N.m J	23·73 pdl ft 0·7375 lb ft 8·85 lb in. $0 \cdot 9478 \times 10^{-3}$ Btu
		kilogramme-metre kilogrammetre[8]			kg.m kgm	232·7 pdl ft 7·233 lb ft 86·796 lb in. $9 \cdot 295 \times 10^{-3}$ Btu
			sthene-metre kilojoule[8]		sn.m kJ	
				dyne-centimetre erg[8]	dyn.cm —	$23 \cdot 73 \times 10^{-7}$ pdl ft
Temperature	degree Kelvin degree Celsius				K C	1·8 °R Temp. °F = 1·8 × (temp. °C) + 32

(1) SI (MKSA) basic units: metre, kilogramme *mass*, second, ampere
The major advantage of this system lies in the fact that 1 newton-metre per second = 1 volt × 1 ampere
(2) MKfS basic units: metre, kilogramme *force*, second
(3) MTS basic units: metre, tonne *mass*, second
(4) CGS basic units: centimetre, gramme *mass*, second
(5) 1 cm = 0·01 m, 1 mm = 0·001 m, 1 μ (micron) = 0·001 mm
(6) 1 t = 1,000 kg, 1 g = 0·001 kg
(7) Newton = the force which, acting upon 1 kg mass, gives it an acceleration of 1 m/s²
Sthene = the force which, acting upon 1 t mass, gives it an acceleration of 1 m/s²
Dyne = the force which, acting upon 1 g mass, gives it an acceleration of 1 cm/s²
1 kg(f) = 9·81 N, 1 sn = 1,000 N, 1 dyn = 10^{-5} N
(8) 1 kgm = 9·81 J, 1 kJ = 1,000 J, 1 erg = 10^{-7} J

Table U2

Quantity, Symbol	*Metric unit*	*System*	*British equivalent*
Volume, V	m^3	MKSA	$35{\cdot}31\ ft^3$
	cm^3	CGS	$0{\cdot}061\ in^3$
	dm^3	—	$35{\cdot}31 \times 10^{-3}\ ft^3$
	or litre (l)		$61{\cdot}02\ in^3$
			0·220 imp. gal
			0·264 US gal
Volume flow, Q	m^3/s	MKSA	$35{\cdot}31\ ft^3/s$
			13,200 imp. gal/min
	cm^3/s	CGS	$13{\cdot}2 \times 10^{-3}$ imp. gal/min
	l/mn	—	0·220 imp. gal/min
Density, ρ	kg/m^3	MKSA	$62{\cdot}43 \times 10^{-3}\ lb/ft^3$
	g/cm^3	CGS	$62{\cdot}43\ lb/ft^3$
			$1{\cdot}94\ slug/ft^3$
			$0{\cdot}0361\ lb/in^3$
Energy, Work done, W	joule, J	MKSA	23·73 pdl ft
	calorie, cal	—	$3{\cdot}967 \times 10^{-3}$ Btu
	(4·1855 J)		
	kilowatt-hour, kWh	—	$8{\cdot}54 \times 10^{7}$ pdl ft
			$2{\cdot}655 \times 10^{6}$ lb ft
			3,412 Btu
Power, H	watt, W	MKSA	23·73 pdl ft/sec
	(joule/s)		0·7375 lb ft/sec
			8·85 lb in /sec
			$1{\cdot}341 \times 10^{-3}$ hp
	kilowatt, kW	MTS	1·341 hp
	cheval-vapeur, ch	—	0.986 hp
Pressure, stress, P, t	pascal, Pa	MKSA	$14{\cdot}50 \times 10^{-5}\ lb/in^2$
	bar	—	$0{\cdot}672 \times 10^{5}\ pdl/ft^2$
	$(= 10^5$ Pa)		$2{,}088\ lb/ft^2$
			$14{\cdot}50\ lb/in^2$
	kgf/cm^2	—	$14{\cdot}22\ lb/in^2$
	kgf/mm^2	—	$14{\cdot}22 \times 10^{2}\ lb/in^2$
Stiffness, R	N/m	MKSA	2·205 pdl/ft
			0·0685 lb/ft
			$5{\cdot}71 \times 10^{-3}$ lb/in
	kgf/mm	—	56 lb/in
Dynamic viscosity, μ	poiseuille, Pl	MKSA	$0{\cdot}672\ pdl\ sec/ft^2$
			$0{\cdot}0209\ lb\ sec/ft^2$
			$14{\cdot}5 \times 10^{-5}\ lb\ sec/in^2$
	poise, Po	CGS	$14{\cdot}5 \times 10^{-6}\ lb\ sec/in^2$
Kinematic viscosity, ν	m^2/s	MKSA	$10{\cdot}764\ ft^2/sec$
			$1{,}550\ in^2/sec$
	stokes, S	CGS	$0{\cdot}155\ in^2/sec$

Table U3. The most common British and American units

Quantity	Units	Metric equivalent: MKSA units	Metric equivalent: miscellaneous units
Length	foot (ft) inch (in.)	0·3048 m $2{\cdot}54 \times 10^{-2}$ m	
Mass	pound* (lb) slug† ounce (oz)	0·4536 kg 14·594 kg	 28·35 g
Force	poundal* (pdl) pound† (lb)	0·13825 N 4·448 N	0·0141 kgf 0·4536 kgf
Energy, Work	poundal-foot (pdl ft) pound-foot (lb ft) pound-inch (lb in.) British thermal unit (Btu)	$4{\cdot}214 \times 10^{-2}$ J 1·356 J 0·113 J 1,055 J	$4{\cdot}297 \times 10^{-3}$ kgm 0·13825 kgm 0·01152 kgm 107·6 kgm 252 cal
Power	horse power (hp)	745·7 W	1·014 ch
Pressure	pound per sq. inch (lb/in.^{2})		0·0689 bar 0·0703 kgf/cm^2
Volume	US gallon (US gal) Imperial gallon (imp. gal)		3·785 l. 4·546 l.
Flow	US gal/min Imp. gal/min		63·09 cm^3/sec 75·77 cm^3/sec
Temperature	Degree Fahrenheit (°F)		$t°\text{F} = \frac{5}{9}(t-32)°\text{C}$

* The poundal is the force which, acting upon a 1 pound mass, gives it a 1 ft/sec^2 acceleration:

$$1 \text{ pdl} = 1 \frac{\text{lb (mass) . ft}}{\text{sec}^2}$$

† The slug is the mass which gets a 1 ft/sec^2 acceleration when submitted to a 1 pound force:

$$1 \text{ slug} = 1 \frac{\text{lb (weight) . sec}^2}{\text{ft}}$$

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